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Singular perturbation of a class of non-convex functionals[☆]

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Abstract

Models involving singular perturbation to a non-convex potential energy play a very important role in describing phase transitions, e.g. the celebrated Cahn–Hillard model where a two-well potential energy functional (i.e., the potential has two zeros) is perturbed by the L^2-norm of the gradient.

Many variants of this model have been studied. In this paper, we perturb a general multi-well energy functional by the L^2-norm of a higher gradient Hessian of arbitrary order and study its $\Gamma(L^1)$-limit. As expected, the limit functional assigns different surface energy densities to interfaces between different phases and computes the total energy.

Keywords: Phase transition; Singular perturbation; Non-convex functionals; Γ-limit

1. Backgrounds

The singular perturbation of non-convex, multi-well integral functionals appears in many fields of material science and engineering, such as the higher strain-gradient theory of plasticity, the magnetic domain theory of ferromagnetic materials, the surface energy theory of crystalline materials, etc. For example, in the classical theory of the phase transition of non-interacting mixture of fluid (see [14,15]), the stable constitution

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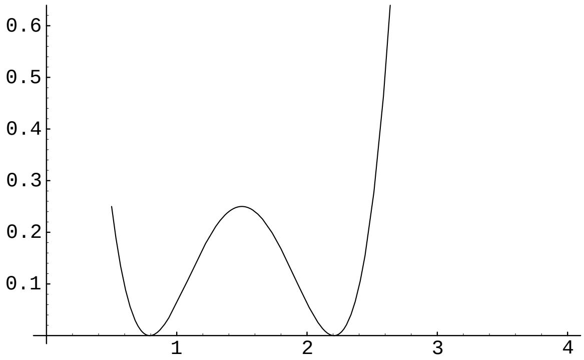

Fig. 1. Picture of $W(u)$ when $d=1,\ r=2$.

of the fluid solves the following problem:

$$\min\left\{\int_\Omega W(u)\,\mathrm{d}x\colon u\in L^1(\Omega;\mathbb{R}^d),\ \int_\Omega u\,\mathrm{d}x=|\Omega|\left(\sum_{i=1}^r\theta_i a_i\right)\right\},$$

where Ω is the domain containing the fluid. Non-negative function W stands for the unit Gibbs free energy, which is related to some properties (e.g. density) of the fluid, and $\{u\colon W(u)=0\}=\{a_1,\dots,a_r\}\subset\mathbb{R}^d$, where $\{a_i\}$ are the value of those properties in stable states (see Fig. 1 for a picture of W when $d=1,\ r=2$). The constraint $\int_\Omega u=\text{constant}$ shows the conservation of those characteristic values. The solution to the above problem shows the stable state distribution of the components of the mixture.

It is easy to see that the solutions to the above problem are far from unique. It is believed that this non-uniqueness is due to the negligence of some small effects in the mathematical model. Following the idea of the celebrated Van Der Waals–Cahn–Hilliard model [15], we try to restore the uniqueness by adding some perturbation terms, which serve as penalties. In general, the perturbation term, which stands for the energy cost of the formation of interfaces between different phases, involves some higher order derivatives. Let

$$J_\varepsilon(u)=\int_\Omega (W(u)+h^2(\varepsilon,x,Du,D^2u,\cdots))\,\mathrm{d}x$$

be the perturbed functional, where $h\to 0$ when $\varepsilon\to 0$. We hope that a proper limit of $J_\varepsilon(u)$ can select a unique and reasonable solution from an infinite number of minimizers of the original problem. It turns out that the so-called "Γ-limit" is the right idea to serve this purpose.

Following the above ideas, many results appeared in the past two decades (see e.g. [2,4,6,9–12,16,17,19]). All these results proved that the Γ-limit of J_ε should be the minimizer of a functional of the following type:

$$\sum_{\substack{i\neq j\\ 1\leqslant i,j\leqslant r}} S_{ij}m_{ij}\ :\ u \text{ is a solution to the un-perturbed problem,}$$

where S_{ij} is the area of the interface—the boundary shared by any two domains, in which u takes a_i and a_j respectively, and m_{ij} is the "unit surface energy" related to that interface.

In this paper, following the ideas of [4,11], we generalize the results to $d > 1, r > 2$, with the perturbation term being $\varepsilon^{2k} \int_\Omega |D^k u|^2 \, \mathrm{d}x$.

2. Preliminaries

2.1. The space of BV functions

Here we list the definition and some properties of BV functions, which we will need in the following sections of this article.

Definition 2.1.1. Let $\Omega \subseteq \mathbb{R}^N$ be open, and $f \in L^1(\Omega)$. Define

$$\int_\Omega |Df| = \sup_{\|g(x)\|_{L^\infty(\Omega;\mathbb{R}^N)} \leqslant 1} \left\{ \int_\Omega f \, \mathrm{Div}\, g \, \mathrm{d}x \right\},$$

where $g = (g_1, \ldots, g_n) \in C_0^1(\Omega; \mathbb{R}^N)$.

Definition 2.1.2. A function $f \in L^1(\Omega)$ is called having bounded variation in Ω, if $\int_\Omega |Df| < \infty$ as defined above. We call the space of all $L^1(\Omega)$ functions which have bounded variation $\mathrm{BV}(\Omega)$.

Definition 2.1.3. Let E be a Borel set in $\mathbb{R}^N, \Omega$ an open set in $\mathbb{R}^N$. We define as follows the perimeter of E in Ω:

$$\mathrm{Per}_\Omega(E) = \int_\Omega |D\chi_E|,$$

where χ_E is the characteristic function of E.

Definition 2.1.4. If a Borel set E has locally finite perimeter, i.e., $\mathrm{Per}_\Omega(E) < +\infty$ for any bounded open set Ω, then E is called a Caccioppoli set.

Let E be a Caccioppoli set, then there exists a subset $\partial^* E$ of ∂E, which has the following property:

For any open set $\Omega \in \mathbb{R}^N$, we have

$$\mathrm{Per}_\Omega(E) = \mathscr{H}^{N-1}(\partial^* E \cap \Omega),$$

where $\mathscr{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure.

Later we will use the following important theorem about BV functions.

Theorem 2.1.5. *For any Caccioppoli set E, there exist a series of C^∞ sets E_j, such that*

$$\int |\chi_E - \chi_{E_j}| \, \mathrm{d}x \to 0, \quad \int |D\chi_{E_j}| \to \int |D\chi_E|.$$

For detailed theory of BV functions and boundary of sets, see e.g. [13].

2.2. Γ-limit

The theory of Γ-limit provides a good frame for the study of the asymptotic behavior of the minimizers of a series of functionals. For the details of this theory, see [8].

Now we state the definition of $\Gamma(L^1)$-limit.

Definition 2.2.1. Suppose $F_n : L^1(\Omega; \mathbb{R}^d) \to [-\infty, \infty]$, $u \in L^1(\Omega; \mathbb{R}^d)$. Define Γ-upper limit and Γ-lower limit as the following:

$$\Gamma(L^1) - \liminf F_n(u) := \inf \left\{ \liminf_{n\to\infty} F_n(u_n) \colon u_n \to u \text{ in}^1(\Omega; \mathbb{R}^d) \right\},$$

$$\Gamma(L^1) - \limsup F_n(u) := \inf \left\{ \limsup_{n\to\infty} F_n(u_n) \colon u_n \to u \text{ in } L^1(\Omega; \mathbb{R}^d) \right\}.$$

If $\Gamma(L^1) - \liminf F_n(u) = \Gamma(L^1) - \limsup F_n(u)$, then the common limit is called the $\Gamma(L^1)$-limit of F_n at u, and is denoted as $\Gamma(L^1) - \lim F_n(u)$.

Moreover, for a set of functionals $F_\varepsilon : L^1(\Omega; \mathbb{R}^d) \to [-\infty, \infty], \varepsilon > 0$, for $u \in L^1(\Omega; \mathbb{R}^d)$, we say $\Gamma(L^1) - \lim F_\varepsilon(u) = F(u)$, if for all $\varepsilon_n \to 0^+, F(u) = \Gamma(L^1) - \lim F_{\varepsilon_n}(u)$.

Here we should state an important observation. i.e., any minimizer u_ε of the functional F_ε is also a minimizer of $\lambda_\varepsilon F_\varepsilon$ for any $\lambda_\varepsilon > 0$. According to this observation, we can say that any minimizer of the Γ-limit of each $\{\lambda_\varepsilon F_\varepsilon\}$ reveals some asymptotic properties of $\{u_\varepsilon\}$. It turns out that there exists a "best" $\{\lambda_\varepsilon\}$, in the sense that the Γ-limit of this $\lambda_\varepsilon F_\varepsilon$ reveals the most.

3. Main result

Consider the following problem:

Let $\Omega \in \mathbb{R}^N$ be a bounded open set, Let the functions $u : \Omega \to \mathbb{R}^d$ and $W : \mathbb{R}^d \to \mathbb{R}$ satisfy the following properties:

(H1) $W(u) \geqslant 0, W(u) = 0 \Leftrightarrow u \in \{a_1, \ldots, a_r\}$, where $a_i \in \mathbb{R}^d$, $i = 1, \ldots, r$.

(H2) $W(u)$ continuous, has no less of order one growth at infinity, i.e., there exist $C, R > 0$, such that when $|u| > R$, $W(u) > C|u| - 1/C$.

We will develop the Γ-limit of the following series of functionals:

$$\mathscr{F}_\varepsilon(u) = \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon^{2k-1} |D^k u|^2 \right) \mathrm{d}x.$$

where $|D^k u|^2 \equiv \sum_{\{i_1,\dots,i_k\}} |\partial^k u/(\partial x_{i_1}\cdots\partial x_{i_k})^2$ with the summation taken over all ordered subsets with cardinal k of $\{1,2,\dots,N\}$.

3.1. Preparative lemmas

Lemma 3.1.1. *Let $\varphi:(0,+\infty)\to\mathbb{R}$ be non-decreasing and convex, J a interval with integer or infinite length, then for any $u\in L^1_{\mathrm{loc}}(J,\mathbb{R}^d)$, $u^{(k)}\in L^1_{\mathrm{loc}}(J,\mathbb{R}^d)$, $(k\geqslant 2,\ k\in\mathbb{N})$, there exists $C=C(k)>0$, such that the following holds, if all the involved integrals are well defined*:

$$\int_J \varphi\left(\frac{|u^{(l)}(t)|}{(1+C)d}\right)\mathrm{d}t$$

$$\leqslant \frac{C}{C+1}\int_J (\varphi(|u|(t))+\varphi(|u^{(k)}(t)|))\,\mathrm{d}t, \quad \forall 1\leqslant l\leqslant k-1.$$

Proof. First we consider the case $d=1$. For $u(t)\in W^{k,1}(0,1)$, we divide $(0,1)$ equally into $2^{l+1}-1$ intervals $I_1,\dots,I_{2^{l+1}-1}$. Fix 2^l points

$$\theta_i^0\in I_{2i-1},\quad i=1,\dots,2^l.$$

Denote θ_i^j to be the point satisfying

$$u^{(j)}(\theta_i^j)=\frac{u(\theta_{2i}^{j-1})-u(\theta_{2i-1}^{j-1})}{\theta_{2i}^{j-1}-\theta_{2i-1}^{j-1}},\quad i=1,\dots,2^{l-j},$$

where $j=1,2,\dots,l$. Given the positions of θ_i^0, the absolute value of each divider has a common lower bound, which depends only on k. Using the mean value theorem, we have: there exists $\eta\in(0,1), C=C(k)>0$, such that

$$|u^{(l)}(\eta)|\leqslant C\sum_{i=1}^{2^l}|u(\theta_i^0)|,\quad i=1,\dots,2^l,$$

then for any $x\in(0,1)$, we have

$$|u^{(l)}(x)|\leqslant C\sum_{i=1}^{2^l}|u(\theta_i^0)|+\int_0^1|u^{(l+1)}(t)|\,\mathrm{d}t.$$

Note that this inequality holds for all $\theta_i^0\in I_{2i-1}$. Then we integrate θ_i^0 on each I_{2i-1}. We have

$$|I_{2i-1}|^{2^l}|u^{(l)}(x)|\leqslant C|I_{2i-1}|^{2^l-1}\sum_{i=1}^{2^l}\int_{I_{2i-1}}|u(\theta_i^0)|\,\mathrm{d}\theta_i^0+|I_{2i-1}|^{2^l}\int_0^1|u^{(l+1)}(t)|\,\mathrm{d}t.$$

Note that

$$\sum_{i=1}^{2^l} \int_{I_{2i-1}} |u(\theta_i^0)| \,\mathrm{d}\theta_i^0 = \sum_{i=1}^{2^l} \int_{I_{2i-1}} |u(t)| \,\mathrm{d}t \leqslant \int_0^1 |u(t)| \,\mathrm{d}t.$$

We have

$$|u^{(l)}(x)| \leqslant C \int_0^1 |u(t)| \,\mathrm{d}t + \int_0^1 |u^{(l+1)}(t)| \,\mathrm{d}t.$$

Applying the above argument to $u^{(l+1)}$, we have:

$$|u^{(l+1)}(x)| \leqslant C \int_0^1 |u(t)| \,\mathrm{d}t + \int_0^1 |u^{(l+2)}(t)| \,\mathrm{d}t.$$

Putting it into the former formula, we have

$$|u^{(l)}(x)| \leqslant C \int_0^1 |u(t)| \,\mathrm{d}t + \int_0^1 |u^{(l+2)}(t)| \,\mathrm{d}t.$$

Repeating the above process, we can find a constant C, depending only on k, such that

$$|u^{(l)}(x)| \leqslant C \int_0^1 |u(t)| \,\mathrm{d}t + \int_0^1 |u^{(k)}(t)| \,\mathrm{d}t$$

holds for any $1 \leqslant l \leqslant k-1$. So we have

$$\frac{1}{C+1} |u^{(l)}(x)| \leqslant \frac{C}{C+1} \int_0^1 |u(t)| \,\mathrm{d}t + \frac{1}{C+1} \int_0^1 |u^{(k)}(t)| \,\mathrm{d}t.$$

Due to the Embedding Theorem, $u^{(l)}$ is continuous. So the above holds almost everywhere.

Because φ is convex and non-decreasing, we have

$$\begin{aligned}
\varphi\left(\frac{1}{C+1} |u^{(l)}(x)|\right) &\leqslant \varphi\left(\frac{C}{C+1} \int_0^1 |u(t)| \,\mathrm{d}t + \frac{1}{C+1} \int_0^1 |u^{(k)}(t)| \,\mathrm{d}t\right) \\
&\leqslant \frac{C}{C+1} \varphi\left(\int_0^1 |u(t)| \,\mathrm{d}t\right) + \frac{1}{C+1} \varphi\left(\int_0^1 |u^{(k)}(t)| \,\mathrm{d}t\right) \\
&\leqslant \frac{C}{C+1} \left[\int_0^1 \varphi(|u(t)|) \,\mathrm{d}t + \int_0^1 \varphi(|u^{(k)}(t)|) \,\mathrm{d}t\right]
\end{aligned}$$

for all $x \in (0,1)$. Integrating on x from 0 to 1, we have

$$\int_0^1 \varphi\left(\frac{|u|^{(l)}(t)}{C+1}\right) \mathrm{d}t \leqslant \frac{C}{C+1} \int_0^1 (\varphi(|u(t)|) + \varphi(|u^{(k)}(t)|)) \,\mathrm{d}t.$$

Dividing J into intervals with length 1, using the above result on each interval, and then adding them together, we prove the case $d = 1$.

For $d \geqslant 2$, let $u(t) = (u_1(t), \ldots, u_d(t))$, we have

$$\begin{aligned}
\int_J \varphi\left(\frac{|u^{(l)}(t)|}{(1+C)d}\right) \mathrm{d}t &\leqslant \int_J \varphi\left(\sum_{i=1}^{d} \frac{|u_i^{(l)}(t)|}{(1+C)d}\right) \mathrm{d}t \\
&\leqslant \frac{1}{d}\sum_{i=1}^{d} \int_J \varphi\left(\frac{|u_i^{(l)}(t)|}{C+1}\right) \mathrm{d}t \\
&\leqslant \frac{C}{(C+1)d}\sum_{1}^{d} \int_J [\varphi(|u_i(t)|) + \varphi(|u_i^{(k)}(t)|)]\,\mathrm{d}t \\
&\leqslant \frac{C}{(C+1)d}\sum_{1}^{d} \int_J [\varphi(|u(t)|) + \varphi(|u^{(k)}(t)|)]\,\mathrm{d}t \\
&= \frac{C}{C+1} \int_J [\varphi(|u(t)|) + \varphi(|u^{(k)}(t)|)]\,\mathrm{d}t. \qquad \square
\end{aligned}$$

Definition 3.1.2. We define auxiliary functions $G_i : \mathbb{R}^{kd} \to \mathbb{R}$, $i \in \{1, \ldots, r\}$, sets $\mathscr{A}_{ij}$, $\tilde{\mathscr{A}}_{ij}$ and constants $m_{ij}, \tilde{m}_{ij}$ for later use:

$$G_i(z_0, \ldots, z_{k-1}) := \inf_g \left\{ \int_0^1 (W(g(t)) + |g^{(k)}(t)|^2)\,\mathrm{d}t \right\},$$

where $g(t) \in C^k([0,1]; \mathbb{R}^d)$, $g(0) = z_0, \ldots, g^{(k-1)}(0) = z_{k-1}$; $g(1) = a_i$, $g^{(l)}(1) = 0$, $l = 1, \ldots, k-1$.

$$\mathscr{A}_{ij} := \{f \in W^{k,2}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^d) : \exists C > 0, \quad f(t) = a_i \quad \text{for } t < -C, \quad f(t) = a_j$$

$$\text{for } t > C, \quad i \neq j\},$$

$$\tilde{\mathscr{A}}_{ij} := \left\{ f \in W^{k,2}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^d) : \lim_{t \to -\infty} f(t) = a_i, \lim_{t \to +\infty} f(t) = a_j \right\},$$

$$m_{ij} := \inf_f \left\{ \int_{\mathbb{R}} (W(f) + |f^{(k)}|^2)\,\mathrm{d}t : f \in \mathscr{A}_{ij} \right\}$$

and

$$\tilde{m}_{ij} := \inf_f \left\{ \int_{\mathbb{R}} (W(f) + |f^{(k)}|^2)\,\mathrm{d}t : f \in \tilde{\mathscr{A}}_{ij} \right\}.$$

Remark 3.1.3. Using polynomials as test functions, we can easily see that when $z_0 \to a_i$ and $z_l \to 0$, $l = 1, \ldots, k-1$, there holds $G_i(z_0, \ldots, z_{k-1}) \to 0$.

Lemma 3..1.4. *For any* $1 \leqslant i, j \leqslant r$, $i \neq j$, *we have* $m_{ij} = \tilde{m}_{ij} > 0$, *and* $\tilde{m}_{ij}$ *is attained.*

Before proving this lemma, we cite a result of Nirenberg in [18].

Theorem 3.1.5. *Let $\Omega \in \mathbb{R}^n$, suppose $u \in L^q(\Omega)$, its mth derivative $D^m u \in L^r(\Omega)$, $1 \leqslant q$, $r \leqslant \infty$. Then for $D^j u$, $0 \leqslant j < m$, the following inequality holds*:

$$|D^j u|_p \leqslant C(|D^m u|_r^{\alpha} |u|_q^{1-\alpha} + |u|_q),$$

where

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\alpha)\frac{1}{q}$$

for all α satisfying $\frac{j}{m} \leqslant \alpha \leqslant 1$. where C depends on $n, m, j, q, r, \alpha, \Omega$. There are two exceptions:

1. *If $j = 0$, $rm < n$, $q = \infty$, then we should suppose further either u tends to 0 at infinity, or $u \in L^{\tilde{q}}$, holds for some finite $\tilde{q} > 0$.*
2. *If $1 < r < \infty$, and $m - j - n/r$ a non-negative integer, then α should satisfy $j/m \leqslant \alpha < 1$.*

Remark 3.1.6. We would like to make the following explanations:

1. The original theorem deals with the case $\Omega = \mathbb{R}^n$, the result we cite here is in fact its fifth remark.
2. There is also a slight change in notations, in [18] the semi-norm $|D^k u|_p$ is defined as the maximum of the $|\cdot|_p$ norms of all jth order derivatives of u, but obviously this change would not affect the result, as long as we take a larger constant C.
3. The above theorem holds for all $p \in \mathbb{R}^1/\{0\}$, as is stated in [18], with the norms for $p < 0$ suitably defined as some Hölder norm. We will not go into the details here since all we need in the following is the $p > 0$ case.
4. Although we will not need it, it is worth mentioning that since for any measurable function $u : \Omega \to \mathbb{R}$, when Ω is bounded, we have

   $$\|u\|_{L^\infty} = \lim_{p \to +\infty} \|u\|_{L^p}$$

   so the case $p = +\infty$ needs no special treatment, just let α tend to some α_0 which causes p to be $+\infty$ would do.

Now we proceed to prove Lemma 3.1.2.

Proof. First we prove $m_{ij} > 0$. Suppose this is not true, that is $m_{ij} = 0$. If f_n is a minimizing sequence, then $f_n^{(k)} \to 0$ in L^2. Using Sobolev Embedding Theorem [1], we have $f_n \in C^{k-1}(\mathbb{R}; \mathbb{R}^d)$.

Let $S = \{y : y \in \mathbb{R}^d, |y - a_i| = |a_i - a_j|/2\}$, then from the definition of $\mathscr{A}_{ij}$ we have

$$\exists M_n > 0, \quad \text{such that} \begin{cases} f_n = a_j, \text{ for } t > M_n, \\ f_n = a_i, \text{ for } t < -M_n. \end{cases}$$

So there exists $t_n \in \mathbb{R}$ such that $f_n(t_n) \in S$. It's easy to see that we can take $t_n = 0$.

Take $\alpha = (j/m)$, $r = 2$, $q = 1$, $m = k$ in theorem 3.1.3, we have $p = 2m/(2m - j)$. So $\|f_n\|_{W^{k,2m/(2m-1)}}$ are uniformly bounded on any interval J. Using again the Embedding

Theorem, we know that there exists a subsequence (still denote as f_n) converges on J under the $W^{k-1,\infty}$ norm to some $f \in W^{k,2m/(2m-1)}_{\text{loc}}(\mathbb{R};\mathbb{R}^d)$, which satisfies $f^{(k)} = 0$ in J, $f(0) \in S$. Now,

$$\begin{aligned} m_{ij} &= \lim_{n\to\infty} \int_{\mathbb{R}} (W(f_n) + |f_n^{(k)}|^2)\,\mathrm{d}t \\ &\geqslant \limsup_{n\to\infty} \int_J (W(f_n) + |f_n^{(k)}|^2)\,\mathrm{d}t \\ &\geqslant \int_J W(f)\,\mathrm{d}t > 0. \end{aligned}$$

This leads to a contradiction. So we have $m_{ij} > 0$.

Next we prove $m_{ij} = \tilde{m}_{ij}$.

Obviously $m_{ij} \geqslant \tilde{m}_{ij}$. Now we prove the opposite direction. Fix $\delta > 0$ and $f \in \tilde{\mathscr{A}}_{ij}$ satisfying $\tilde{m}_{ij} + \delta > \int_{\mathbb{R}} (W(f) + |f^{(k)}|^2)\,\mathrm{d}t$. Now we will construct a sequence of functions in $\mathscr{A}_{ij}$, which approaches $\tilde{m}_{ij} + \delta$.

Fix $\tau < |a_j - a_i|/2$, consider a non-decreasing convex function $\varphi : \mathbb{R} \to [0, +\infty)$, satisfying

$$\varphi(t) < t^2,\ \forall t \in \mathbb{R},$$

$$\varphi(|y|) < W(y + a_j),\ \forall y \in B(0,\tau) \setminus \{0\} \subset \mathbb{R}^d,$$

$$\varphi(t) = 0 \Leftrightarrow t = 0.$$

It is easy to prove that such φ exists.

Take $R > 0$, such that $|f(t) - a_j| < \tau$, $\forall t > R$. Using Lemma 3.1 on $f - a_j$ we have

$$\int_R^{+\infty} \varphi\left(\frac{|f^{(l)}|}{(C+1)d}\right) < \frac{C}{C+1} \int_R^{+\infty} (W(f) + |f^{(k)}|^2) \leqslant \frac{C}{C+1}(\tilde{m}_{ij} + \delta),$$

thus $\sum_{l=1}^{k-1} \varphi(|f^{(l)}|/(C+1)d)$ is integrable on $(R, +\infty)$. So there exist $x_n \to +\infty$ such that

$$\lim_{n\to\infty} \left[\sum_{l=1}^{k-1} \varphi\left(\frac{|f^{(l)}(x_n)|}{(C+1)d}\right) + |f(x_n) - a_j|\right] = 0.$$

Since φ is monotone, and is 0 only at $x = 0$, we have

$$\sum_{l=1}^{k-1} |f^{(l)}(x_n)| + |f(x_n) - a_j| \to 0.$$

Similarly, we can prove that there exist $y_n \to -\infty$ such that

$$\sum_{l=1}^{k-1} |f^{(l)|(y_n)}| + |f(y_n) - a_i| \to 0.$$

Take $g_m(t)=\tilde{g}_m(t-x_m)$, $h_m(t)=\tilde{h}_m(t-y_m+1)$, where $\tilde{g}_m$, $\tilde{h}_m$ are admissible functions to $G_j(f(x_m),\dots,f^{(k-1)}(x_m))$ and $G_i(f(y_m),\dots,f^{(k-1)}(y_m))$, respectively, satisfying

$$\int_0^1 (W(\tilde{g}_m)+|\tilde{g}_m^{(k)}|^2)\,\mathrm{d}t \leqslant G_j(f(x_m),\dots,f^{(k-1)}(x_m))+\delta,$$

$$\int_0^1 (W(\tilde{h}_m)+|\tilde{h}_m^{(k)}|^2)\,\mathrm{d}t \leqslant G_i(f(y_m),\dots,f^{(k-1)}(y_m))+\delta.$$

Let

$$\tilde{f}_m(t)=\begin{cases} a_j & t\geqslant x_m+1,\\ g_m(t) & t\in[x_m,x_m+1],\\ f(t) & t\in(y_m,x_m),\\ h_m(t) & t\in[y_m-1,y_m],\\ a_i & t\leqslant y_m-1,\end{cases}$$

we then have

$$\begin{aligned}\tilde{m}_{ij}+\delta &> \int_{\mathbb{R}}(W(f)+|f^{(k)}|^2)\,\mathrm{d}t\\ &> \int_{x_m}^{y_m}(W(f)+|f^{(k)}|^2)\,\mathrm{d}t\\ &= \int_{\mathbb{R}}(W(\tilde{f}_m)+|\tilde{f}_m^{(k)}|^2)\,\mathrm{d}t-\int_{x_m}^{x_m+1}(W(g_m)+|g_m^{(k)}|^2)\,\mathrm{d}t\\ &\quad -\int_{y_m-1}^{y_m}(W(h_m)+|h_m^{(k)}|^2)\,\mathrm{d}t\\ &\geqslant m_{ij}-G_j(f(x_m),\dots,f^{(k-1)}(x_m))-G_i(f(y_m),\dots,f^{(k-1)}(y_m))\\ &\to m_{ij}.\end{aligned}$$

Thus we proved $\tilde{m}_{ij}\geqslant m_{ij}$. Finally, we prove that the minimum is attained.

Suppose $\{f_n\}$ is a minimizing sequence of $\tilde{m}_{ij}$. We can safely assume that $f_n(0)\in S$, and $\{f_n\}$ converges to $f\in W^{k,2}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^d)$ in $W^{k-1,\infty}$. It is easy to see that $f(0)\in S$. If f is admissible, due to Fatou's Lemma and the convexity of L^2 norm, f is a minimizer. Now we prove that f is indeed admissible, i.e., $f\in\tilde{\mathscr{A}}_{ij}$.

Let $L=\{l\in\mathbb{R}^d \mid l$ is a limit point of $f(t)$ as $t\to+\infty\}$, since $W(f)$ is integrable, we know that either a_i or a_j, or both, are in L. No harm to let $a_j\in L$. If there is another $l\neq a_j$ in L, we can suppose $l\neq a_i$. (If $a_i\in L$, then using the continuity of f_n, and the compactness of S, we know that there exists a $l\in S$, such that $l\in L$.) Consider two increasing sequences $\{x_i\}$, $\{z_i\}$, satisfying $x_{i+1}-x_i\geqslant 3$, $z_i\in[x_i+1,x_{i+1}-1]$,

$f(x_i) \to a_j$, $f(z_i) \to l$. For any $0 < \delta < \min\{|l - a_i|, |l - a_j|\}$ we define

$$\hat{m} = \inf\left\{\int_x^y (W(g) + |g^{(k)}|^2)\,\mathrm{d}x\colon\ y - x \geqslant 3, \exists z \in [x+1, y-1], |g(z) - l| \leqslant \delta\right\},$$

where $g \in W^{k,2}((x,y);\mathbb{R}^d)$. Now we prove $\hat{m} = 0$.

Suppose the reverse is true, i.e., $\hat{m} > 0$. From the definition of z_n there must exist a n_0, such that $\forall n > n_0$, $|f(z_n) - l| < \delta$, so

$$\int_{\mathbb{R}} (W(f) + |f^{(k)}|^2)\,\mathrm{d}x \geqslant \sum_{n_0}^{+\infty} \int_{x_i}^{x_{i+1}} (W(f) + |f^{(k)}|^2)\,\mathrm{d}x = +\infty.$$

A contradiction!

Suppose $g_n \in W^{k,2}((x_n, y_n);\mathbb{R}^d)$ minimize $\hat{m}$, no harm to suppose $z_n = 0$, so $x_n \leqslant -1$, $y_n \geqslant 1$. Then there exists a subsequence (denoted again by $\{g_n\}$), such that $g_n \to g$ in $W^{k-1,\infty}(-1,1)$. Now we have $\int_{-1}^1 W(g) + |g^{(k)}|^2\,\mathrm{d}x = 0$ and $|g(0) - l| < \delta$ at the same time, a contradiction!

So, when $t \to +\infty$, $f_n \to a_j$.

Using similar induction, if $a_i \in L$, then $f(t) \to a_i, t \to +\infty$. For $t \to -\infty$ we can argue similarly. Now we only need to show that the following does not happen: $f(t) \to a_i, t \to \pm\infty$ or $f(t) \to a_j, t \to \pm\infty$.

Suppose $\lim_{\pm\infty} f(t) = a_i$, no harm to assume $f(0) \in S$. Then there exists $x_n \to +\infty$, such that $\sum_{i=1}^{k-1} |f^{(l)}(x_n)| + |f(x_n) - a_i| \to 0$, so we can find a subsequence of the minimizing sequence $\{f_n\}$(denoted again by $\{f_n\}$), such that $\sum_{i=1}^{k-1} |f_n^{(l)}(x_n)| + |f_n(x_n) - a_i| \to 0$ and $f_n(0) \to f(0)$. Now we have

$$\begin{aligned}
m_{ij} &= \int_{\mathbb{R}} (W(f) + |f^{(k)}|^2)\,\mathrm{d}t \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} (W(f_n) + |f_n^{(k)}|)\,\mathrm{d}t \\
&= \lim_{n\to\infty} \left(\int_{-\infty}^{x_n} (W(f_n) + |f_n^{(k)}|^2)\,\mathrm{d}t + \int_{x_n}^{+\infty} (W(f_n) + |f_n^{(k)}|^2)\,\mathrm{d}t\right) \\
&\geqslant \limsup_{n\to\infty} \left(\int_{-\infty}^{x_n} (W(f_n) + |f_n^{(k)}|^2) + \tilde{m}_{ij} - G_i(f_n(x_n), \ldots, f_n^{(k-1)}(x_n))\right) \\
&= \limsup_{n\to\infty} \int_{-\infty}^{x_n} (W(f_n) + |f_n^{(k)}|^2) + \tilde{m}_{ij}.
\end{aligned}$$

Since $f_n(0) \to f(0) \in S$, it is easy to know that the first term on the right-hand side is greater than 0. A contradiction. $\square$

Remark 3.1.7. Energy balance: Suppose f is the minimizer in the above theorem, define $f_\lambda(t) = f(\lambda t)$, then

$$\int_{\mathbb{R}} (W(f) + |f^{(k)}|^2)\,\mathrm{d}t \leqslant \int_{\mathbb{R}} (W(f_\lambda) + |f_\lambda^{(k)}|^2)\,\mathrm{d}t$$

holds for all $\lambda\in\mathbb{R}$, let $I(\lambda)=\int_{\mathbb{R}}(W(f_\lambda)+|f^{(k)}|^2)\,\mathrm{d}t$, then $\lambda=1$ is a critical point of $I(\lambda)$, so we have

$$\int_{\mathbb{R}} W(f)=(2k-1)\int_{\mathbb{R}}|f^{(k)}|^2.$$

Thus we see that in the transition layer between two phase domains, the Gibbs free energy and the surface energy can be roughly said "equal".

3.2. Compactness result

In this section we will establish the compactness of the minimizing sequence.

Theorem 3.2.1. *Let I be a bounded open interval in $\mathbb{R}$, if $u_\varepsilon\in W^{k,2}(I;\mathbb{R}^d)$ satisfies $\liminf_{\varepsilon\to 0^+}\mathscr{F}_\varepsilon(u_\varepsilon)<+\infty$, then there exists a subsequence $\{u_{\varepsilon_n}\}$ and $u\in \mathrm{BV}(I;\{a_1,a_2,\dots,a_r\})$, such that $u_{\varepsilon_n}\to u$ in $L^1(I;\mathbb{R}^d)$.*

Proof. Let $\liminf\mathscr{F}_\varepsilon(u_\varepsilon)=K<+\infty$, and it is subsequence $\{u_{\varepsilon_n}\}$ satisfy $\lim\mathscr{F}_{\varepsilon_n}(u_{\varepsilon_n})=K$, then it is easy to know that $\{u_{\varepsilon_n}\}$ satisfies the condition of the theorem in [5], so there is a subsequence (denoted again by u_{ε_n}) and a Young measure $v_t(y)$, such that for any continuous function f, we have

$$f(t,u_n(t))\overset{*}{\rightharpoonup}\bar{f}=\int_{\mathbb{R}^d}f(t,y)\,\mathrm{d}v_t(y)\ \text{in}\ L^\infty(I).$$

Let $f(y)=\min\{W(y),1\}$, then it is easy to see that

$$v_t(y)=\sum_{i=1}^{r}\theta_i(t)\delta_{y=a_i},$$

Next we prove $\theta_i\in\{0,1\}$, $i=1,\dots,r$.

Define

$$X_{ij}=\left\{t\in I;\frac{1}{2\delta}\int_{t-\delta}^{t+\delta}\theta_k(s)\,\mathrm{d}s\in(0,1),\quad k=i,j,\ \ \forall\delta>0\right\},$$

we will prove the finiteness of the above set.

Suppose there are $l>1$ distinct points $\{s_1,s_2,\dots,s_l\}$ in X_{ij}, and set $\delta_0=\min\{|s_m-s_{m+1}|\}$. Take $\delta_1<\delta_0/2$, such that for all $\delta<\delta_1$ and all $m\in\{1,\dots,l\}$,

$$\int_{s_m-\delta}^{s_m+\delta}\theta_i(s)\,\mathrm{d}s>0,\quad \int_{s_m-\delta}^{s_m+\delta}\theta_j(s)\,\mathrm{d}s>0.$$

Fix $0<\eta<|a_j-a_i|/2$, suppose φ_η, ψ_η and $\gamma_\eta:\mathbb{R}^d\to\mathbb{R}$ are smooth functions, satisfying

$$\operatorname{supp}\varphi_\eta\subset B(a_i,\eta),\quad \varphi_\eta(a_i)=1,$$

$$\operatorname{supp}\psi_\eta\subset B(0,\eta),\quad \psi_\eta(0)=1,$$

$$\operatorname{supp}\gamma_\eta\subset B(a_j,\eta),\quad \gamma_\eta(a_j)=1.$$

From the uniform boundedness of $\mathscr{F}_{\varepsilon_n}(u_{\varepsilon_n})$, we have the uniform boundedness of u_{ε_n} and $\varepsilon_n^{(2k-1)/2}u_{\varepsilon_n}^{(k)}$ in $L^1(I)$ and $L^2(I)$, respectively. For any $i\in\{1,\dots,k-1\}$, Let $n_i=2i$, and take $\alpha=i/k$, $r=2$, $q=1$ in Theorem 3.1.3, we have $\varepsilon_n^{n_i}u_{\varepsilon_n}^{(i)}\to 0$ in $L^1(I)$, so $\psi_\eta(\varepsilon_n^{n_i}u_{\varepsilon_n}^{(i)})\to\psi_\eta(0)$ in $L^1(I)$. Then from the theorem in [5], $\varphi_\eta(u_{\varepsilon_n})$ converges weak $*$ in $L^\infty(I)$ to $\theta_i\varphi_\eta(a_i)+\theta_j\varphi_\eta(a_j)=\theta_i$, and $\gamma_\eta(u_{\varepsilon_n})$ converges weak $*$ in $L^\infty(I)$ to θ_j. It is easy to see that for each $m\in\{1,\dots,l\}$ and each n, there exist $x_{n,m}^+,x_{n,m}^-\in(s_m-\delta_1,s_m+\delta_1)$, $x_{n,m}^-<x_{n,m}^+$, such that

$$u_{\varepsilon_n}(x_{n,m}^+)\in B(a_j,\eta),\quad u_{\varepsilon_n}(x_{n,m}^-)\in B(a_i,\eta),$$

$$|\varepsilon_n^{n_i}u_{\varepsilon_n}^{(i)}(x_{n,m}^+)|<\eta,\quad |\varepsilon_n^{n_i}u_{\varepsilon_n}^{(i)}(x_{n,m}^-)|<\eta.$$

Define

$$g_{n,m}(t)=\hat{g}_{n,m}\left(t-\frac{x_{n,m}^+}{\varepsilon_n}\right),\quad h_{n,m}(t)=\hat{h}_{n,m}\left(t-\frac{x_{n,m}^-}{\varepsilon_n}+1\right),$$

where $\hat{g}_{n,m}$, $\hat{h}_{n,m}$ are, respectively, the admissible functions of $G_j(u_{\varepsilon_n}(x_{n,m}^-),\varepsilon_n^{n_1}u'_{\varepsilon_n}(x_{n,m}^-),\dots\varepsilon^{n_{k-1}}u_\varepsilon^{(k-1)}(x_{n,m}^-))$ and $G_i(u_{\varepsilon_n}(x_{n,m}^+),\varepsilon_n^{n_1}u'_{\varepsilon_n}(x_{n,m}^+),\dots,\varepsilon^{n_{k-1}}u_{\varepsilon_n}^{(k-1)}(x_{n,m}^+))$, satisfying

$$\int_0^1(W(\hat{h}_{n,m})+|\hat{h}_{n,m}^{(k)}|^2)<G_i(u_{\varepsilon_n}(x_{n,m}^+),\varepsilon_n^{n_1}u'_{\varepsilon_n}(x_{n,m}^+),\dots,\varepsilon^{n_{k-1}}u_{\varepsilon_n}^{(k-1)}(x_{n,m}^+))+\varepsilon_n,$$

$$\int_0^1(W(\hat{g}_{n,m})+|\hat{g}_{n,m}^{(k)}|^2)<G_j(u_{\varepsilon_n}(x_{n,m}^-),\varepsilon_n^{n_1}u'_{\varepsilon_n}(x_{n,m}^-),\dots,\varepsilon^{n_{k-1}}u_\varepsilon^{(k-1)}(x_{n,m}^-))+\varepsilon_n.$$

Now change u_{ε_n} as follows:

$$v_{n,m}(t)=\begin{cases} a_j & t\geqslant\frac{x_{n,m}^+}{\varepsilon_n}+1,\\ g(t) & t\in[\frac{x_{n,m}^+}{\varepsilon_n},\frac{x_{n,m}^+}{\varepsilon_n}+1],\\ u_{\varepsilon_n}(\varepsilon_n t) & t\in[\frac{x_{n,m}^-}{\varepsilon_n},\frac{x_{n,m}^+}{\varepsilon_n}],\\ h(t) & t\in[\frac{x_{n,m}^-}{\varepsilon_n}-1,\frac{x_{n,m}^-}{\varepsilon_n}],\\ a_i & t\leqslant\dfrac{x_{n,m}^-}{\varepsilon_n}-1. \end{cases}$$

From the value of n_i we know that $v_{n,m}\in W_{\mathrm{loc}}^{k,2}(\mathbb{R})$, so

$$K\geqslant\liminf_{\varepsilon_n\to0^+}\sum_{m=1}^{l}\int_{x_{n,m}^-}^{x_{n,m}^+}\left(\frac{1}{\varepsilon_n}W(u_{\varepsilon_n})+\varepsilon_n^{2k-1}|u_{\varepsilon_n}^{(k)}|^2\right)\mathrm{d}t$$

$$=\liminf_{\varepsilon_n\to0^+}\sum_{m=1}^{l}\int_{\frac{x_{n,m}^-}{\varepsilon_n}}^{\frac{x_{n,m}^+}{\varepsilon_n}}(W(v_{n,m})+|v_{n,m}^{(k)}|^2)\,\mathrm{d}t$$

$$\geqslant \min\{m_{ij}\}l - \limsup_{\varepsilon_n \to 0^+} \sum_{m=1}^{l} [G_j(u_{\varepsilon_n}(x^-_{n,m}), \varepsilon_n^{n_1} u_{\varepsilon_n}(x^-_{n,m}), \cdots)$$

$$+ G_i(u_{\varepsilon_n}(x^+_{n,m}), \varepsilon_n^{n_1} u_{\varepsilon_n}(x^+_{n,m}), \cdots)].$$

Let $\eta \to 0^+$, we have $K \geqslant \min\{m_{ij}\}l$, and the finiteness of X follows.
So we have $u \in \mathrm{BV}$ and $u_{\varepsilon_n} \to u$ in L^1. □

Theorem 3.2.2. *If* $u_\varepsilon \in W^{k,2}(\Omega; \mathbb{R}^d)$ *satisfy* $\liminf_{\varepsilon \to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon) \leqslant +\infty$, *then there exists a subsequence* $\{u_{\varepsilon_n}\}$ *and* $u \in \mathrm{BV}(\Omega; \{a_1, \ldots, a_r\})$, *such that*

$$u_{\varepsilon_n} \to u \text{ in } L^1(\Omega; \mathbb{R}^d).$$

First we introduce the concept of δ-closeness and a lemma [3,11]:

Definition 3.2.3. Two function sequences $\{u_\varepsilon\}$, $\{v_\varepsilon\}$ are called δ-close, if $\|u_\varepsilon - v_\varepsilon\| < \delta$.

We take $u(y,z): I \times J \to \mathbb{R}$ as an example to define the "slice function" as follows, where I, J are open intervals. For any $y \in I$ we define $u^y(z) := u(y,z)$, and for any $z \in J$ we define $u^z(y) := u(y,z)$, then we call u^y and u^z the one-dimensional slices of u.

Lemma 3.2.4. *Suppose* $\{u_n\}$ *are equiintegrable* [7], *and for any* $\delta > 0$ *there exists sequences* $\{v_n\}$, $\{w_n\}$ δ*-close to* $\{u_n\}$, *and* $\{v_n^y\}$ *precompact for almost every* $y \in I$ *in* $L^1(J; \mathbb{R}^d)$, $\{w_n^z\}$ *precompact for almost every* $z \in J$ *in* $L^1(I; \mathbb{R}^d)$. *Then* $\{u_n\}$ *is precompact in* $L^1(\Omega; \mathbb{R}^d)$.

Now we proceed to prove the theorem.

Proof. No harm to assume $N = 2$. Assume first that $\Omega = I \times J$, where I, J are open intervals. We define the following one-dimensional functional $\mathscr{F}^1_\varepsilon(u, A)$:

$$\mathscr{F}^1_\varepsilon(u,A) := \begin{cases} \int_A \left(\frac{W(u)}{\varepsilon} + \varepsilon^{2k-1}|u^{(k)}|^2\right) \mathrm{d}t & \text{if } u \in W^{k,2}(A; \mathbb{R}^d), \\ +\infty & \text{if } u \in L^1(A; \mathbb{R}^d) \setminus W^{k,2}(A; \mathbb{R}^d), \end{cases}$$

where $u \in L^1(A; \mathbb{R}^d)$ and A is an open interval.

Since $u \in W^{k,2}(\Omega; \mathbb{R}^d)$, by approximating it with C^∞ functions, we know that for almost every $y \in I$, $u^y \in W^{k,2}(J)$, and for almost every $z \in J$, $u^z \in W^{k,2}(I)$. Furthermore,

$$\frac{\partial^k u}{\partial z^k}(x) = \frac{\mathrm{d}^k u^y}{\mathrm{d} z^k}(z), \quad \frac{\partial^k u}{\partial y^k}(x) = \frac{\mathrm{d}^k u^z}{\mathrm{d} y^k}(y), \quad \text{for a.e. } x \in \Omega.$$

From this we can easily see that

$$\mathscr{F}_\varepsilon(u) \geqslant \int_I \mathscr{F}^1_\varepsilon(u^y, J)\, \mathrm{d}y, \quad \mathscr{F}_\varepsilon(u) \geqslant \int_J \mathscr{F}^1_\varepsilon(u^z, I)\, \mathrm{d}z.$$

Now consider a family of functions $\{u_\varepsilon\}$, satisfying $\mathscr{F}_\varepsilon(u_\varepsilon) < C < +\infty$. Thus $\int_\Omega W(u_\varepsilon)\,\mathrm{d}x < C\varepsilon$, then we have $W(u_\varepsilon) \to 0$ in L^1. So from (H2) (see the beginning of this section) we get the equiintegrability of u_ε. Now fix $\delta > 0$, take $\delta' \in (0, \delta)$ satisfying

$$|S| < \delta'|J| \to \sup_{\varepsilon>0} \int_S (|u_\varepsilon(x)| + |a_1|)\,\mathrm{d}x < \delta.$$

For $\varepsilon > 0$, we define as follows $v_\varepsilon : \Omega \to \mathbb{R}^d$:

$$v_\varepsilon^y(z) := \begin{cases} u_\varepsilon^y(z) = u_\varepsilon(y,z) & \text{if } \mathscr{F}_\varepsilon^1(u_\varepsilon^y, J) < C/\delta' \\ a_1 & \text{elsewhere.} \end{cases}$$

Let $Z_\varepsilon := \{x : v_\varepsilon^y \neq u_\varepsilon^y\}$. Since

$$C > \sup_{\varepsilon>0} \int_I \mathscr{F}_\varepsilon^1(u_\varepsilon^y, J)\,\mathrm{d}y,$$

we have

$$|Z_\varepsilon| \leqslant |\{\mathscr{F}_\varepsilon^1(u_\varepsilon^y, J) > C/\delta'\}| \leqslant \frac{\delta'}{C} \int_I \mathscr{F}_\varepsilon^1(u_\varepsilon^y, J)\,\mathrm{d}y \leqslant \delta',$$

so

$$\|u_\varepsilon - v_\varepsilon\|_1 < \int_{Z_\varepsilon \times J} |u_\varepsilon(x) - a_1|\,\mathrm{d}x < \int_{Z_\varepsilon \times J} (|u_\varepsilon(x)| + |a_1|) < \delta$$

holds for any $\varepsilon > 0$. We have that $\{v_\varepsilon\}$ and $\{u_\varepsilon\}$ are δ-close. We can easily get the precompactness of $\{v_\varepsilon^y\}$ as in [7]. Similarly, we can construct $\{w_\varepsilon\}$ δ-close to $\{u_\varepsilon\}$, and for any $z \in J$, $\{w_\varepsilon^z\}$ is precompact in $L^1(I; \mathbb{R}^d)$. Using Lemma 3.2.4, we have the precompactness of u_ε in $L^1(\Omega, \mathbb{R}^d)$. □

3.3. The proof of Γ-limit

Proposition 3.3.1. *If $u_\varepsilon \in W^{k,2}(I; \mathbb{R}^d)$, $\liminf_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon) < +\infty$, then there exists a subsequence $\{u_{\varepsilon_n}\}$ and $u \in \mathrm{BV}(I; \{a_i\})$, such that $u_{\varepsilon_n} \to u$ in $L^1(I; \mathbb{R}^d)$, and moreover,*

$$\liminf_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon) \geqslant \sum_{i,j} m_{ij} \operatorname{Per}_I(\{u = a_i\} \cap \{u = a_j\}).$$

Proof. The first half of the theorem has already been proved in the above section, and the rest is just an easy corollary of the proof of the theorem in the last section. □

Theorem 3.3.2. *If $u \in \mathrm{BV}(I; \{a_1, \ldots, a_r\})$, then*

$$\Gamma - \limsup_{\varepsilon\to 0+} \mathscr{F}_\varepsilon(u) \leqslant \sum_{i,j} m_{ij} \operatorname{Per}_I(\{u = a_i\} \cap \{u = a_j\}).$$

Proof. Denote the jump point of u by $S(u) = \{s_1, \ldots, s_l\} \subset I = (\alpha, \beta)$, $\alpha < s_1 < \cdots < s_l < \beta$. Let $\delta_0 = \min\{s_{j+1} - s_j\colon j = 0, \ldots, l\}$, where $s_0 = \alpha$, $s_{l+1} = \beta$, and $I_i = [s_{i-1} + s_i/2, s_i + s_{i+1}/2]$, $i = 1, \ldots, l$. Fix $\delta \in (0, \delta_0)$, and take $f_{ij} \in \mathscr{A}_{ij}$ satisfying

$$\int_{\mathbb{R}} (W(f_{ij}) + |f_{ij}^{(k)}|^2)\,\mathrm{d}t \leqslant m_{ij} + \delta.$$

Take $\varepsilon_n \to 0^+$, and n sufficiently large, such that $\delta_0/2\varepsilon_n > \max M_{ij}$, where M_{ij} is the constant in the definition of $\mathscr{A}_{ij}$.

Define

$$u_n(t) = \begin{cases} f_{ij}((t-s_i)/\varepsilon_n) & t \in [(s_{i-1}+s_i)/2, (s_i+s_{i+1})/2], [u](s_i) = a_j - a_i, \\ f_{ij}(-(t-s_i)/\varepsilon_n) & t \in [(s_{i-1}+s_i)/2, (s_i+s_{i+1})/2], [u](s_i) = a_i - a_j, \\ u(t) & \text{elsewhere}, \end{cases}$$

where $[u](s_i):=u(s_i) - u(s_{i-1})$. Now we have

$$\limsup_{n\to\infty} \mathscr{F}_\varepsilon(u_n) \leqslant \sum_{i,j} m_{ij} \operatorname{Per}_I(\{u = a_i\} \cap \{u = a_j\}) + l\delta.$$

Take $\delta_n \to 0^+$, and construct a $\{u_n\}$ for each δ_n, we can prove the desired result. $\square$

Theorem 3.3.3. *Suppose* $u \in L^1(\Omega; \mathbb{R}^d)$. *If* $\Gamma - \liminf_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u) < +\infty$, *then* $u \in \mathrm{BV}(\Omega; \{a_1, \ldots, a_r\})$, *and*

$$\Gamma - \liminf_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u) \geqslant \sum_{1\leqslant i<j\leqslant r} m_{ij} \mathscr{H}^{N-1}(\partial^* S_i \cap \partial^* S_j \cap \Omega).$$

Before we prove this theorem, some preparation is needed.

Define

$$\mathscr{A}_{\xi_1,\xi_2} := \{g(t) \mid g \in W^{k,2}_{\mathrm{loc}}(\mathbb{R}), \exists M>0,\ g(t)=\xi_2 \text{ for } t>M;\ g(t)=\xi_1 \text{ for } t<-M\}$$

and the geodesic distance

$$d(\xi_1, \xi_2) := \inf\left\{ \int_{\mathbb{R}} (W(r(t)) + |r^{(k)}(t)|^2)\,\mathrm{d}t, r(t) \in \mathscr{A}_{\xi_1,\xi_2} \right\}.$$

We notice that $d(a_i, a_j) = m_{ij}$ with the latter defined by Definition 3.1.2. Then we define $\varphi_i(\xi) = d(\xi, a_i)$ as the distance function to a_i.

Suppose μ, v are two positive regular Borel measure, we follow [4] by defining as follows their maximum $\mu \vee v$:

$$(\mu \vee v)(A) = \sup\{\mu(A') + v(A'') : A' \cap A'' = \phi,\ A' \cup A'' \subset A,$$
$$A', A'' \text{ is an open set in } \Omega\},$$

where $A \in \Omega$ is an arbitrary open set.

Suppose that $u(x)$ makes $W(u(x)) = 0$ a.e. in Ω, then $u(x) = \sum_{i=1}^r a_i \chi_{S_i}(x)$, where $S_1, \ldots, S_r \subset \Omega$ with no intersection between any two of them, and $|\Omega \setminus \cup_1^r S_i| = 0$. Now we cite a lemma from [4].

Lemma 3.3.4. *Suppose* $\varphi_i \circ u \in \mathrm{BV}(\Omega)$, *denote* μ_i *to be the following Borel measure*: $\mu_i : E \to \int_E |D(\varphi_i \circ u)|$, *then we have the fact that* $\operatorname{Per}_\Omega(S_i) < +\infty$ *holds for* $i = 1, \ldots, r$, *and*

$$\left(\bigvee_{i=1}^r \mu_i\right)(\Omega) = \sum_{1\leqslant i<j\leqslant r} d(a_i, a_j) \mathscr{H}^{N-1}(\partial^* S_i \cap \partial^* S_j \cap \Omega).$$

By the definition of $d(a_i, a_j)$, we have

$$\left(\bigvee_{i=1}^{r} \mu_i\right)(\Omega) = \sum_{1 \leqslant i < j \leqslant r} m_{ij}\mathscr{H}^{N-1}(\partial^* S_i \cap \partial^* S_j \cap \Omega).$$

Now we are going to prove the theorem.

Proof. Suppose $\varepsilon_n \to 0^+$, $u_n \to u$ in $L^1(\Omega; \mathbb{R}^d)$ and $\mathscr{F}_{\varepsilon_n}(u_n)$ converges to $\Gamma - \liminf_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u) < +\infty$. Fix an unit vector $v \in S^{N-1}$, we take no danger to suppose that $u_n|_{L_{y,v}\cap\Omega} \to u|_{L_{y,v}\cap\Omega}$ in $L^1(L_{y,v} \cap \Omega)$ holds for almost every $L_{y,v} := \{y + sv : s \in \mathbb{R}\}$ and $y \in \mathbb{R}^N$. Let

$$u_n^{y,v}(t) := u_n(y + tv) \quad \text{for } \mathscr{H}^{N-1} \text{ a.e. } y \in v^\perp.$$

From the above one-dimensional result, we know that for any $i \in \{1, \ldots, k\}$,

$$|D(\varphi_i \circ u)^{y,v}(L_{y,v} \cap \Omega)| \leqslant \liminf_{n\to\infty} \int_{L_{y,v}\cap\Omega} \left(\frac{W(u_n^{y,v})}{\varepsilon_n} + \varepsilon_n^{2k-1}\left|\frac{\mathrm{d}^k u_n^{y,v}}{\mathrm{d}t^k}\right|\right) \mathrm{d}t.$$

Let $v_i = \varphi_i \circ u$, then from Fatou's lemma and the slicing property of BV functions we have

$$\begin{aligned}
|D(v_i)|(\Omega) &= \int_{y\in v^\perp} |Dv_i^{y,v}|(L_{y,v} \cap \Omega)\,\mathrm{d}\mathscr{H}^{N-1}(y) \\
&\leqslant \int_{y\in v^\perp} \liminf_{n\to\infty} \int_{L^{y,v}\cap\Omega} \left(\frac{W(u_n^{y,v})}{\varepsilon_n} + \varepsilon_n^{2k-1}\left|\frac{\mathrm{d}^k u_n^{y,v}}{\mathrm{d}t^k}\right|\right) \mathrm{d}t\,\mathrm{d}\mathscr{H}^{N-1} \\
&\leqslant \liminf_{n\to\infty} \int_{y\in v^\perp} \int_{L_{y,v}\cap\Omega} \left(\frac{W(u_n)}{\varepsilon_n} + \varepsilon_n^{2k-1}|D^k u_n|^2\right) \mathrm{d}t\,\mathrm{d}\mathscr{H}^{N-1} \\
&= \liminf_{n\to\infty} \int_\Omega \left(\frac{W(u_n)}{\varepsilon_n} + \varepsilon^{2k-1}|D^k u_n|^2\right) \mathrm{d}x \\
&= \Gamma - \liminf_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u).
\end{aligned}$$

From this we can see that $\varphi_i \circ u \in \mathrm{BV}(\Omega)$. We know from the definition of $\vee$ that $\bigvee_{i=1}^r \mu_i$ is the minimum of all measures that are bigger than all μ_i, thus

$$\bigvee_{i=1}^{k} v_i \leqslant \Gamma - \liminf \mathscr{F}_\varepsilon(u).$$

Now we know from Proposition 3.4 that

$$\Gamma - \liminf_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u) \geqslant \sum_{1\leqslant i<j\leqslant r} m_{ij}\mathscr{H}^{N-1}(\partial^* S_i \cap \partial^* S_j \cap \Omega). \qquad \square$$

Theorem 3.3.5. *For any $u \in \mathrm{BV}(\Omega; \{a_1, \ldots, a_r\})$, we have*

$$\Gamma - \limsup_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u) \leqslant \sum_{1\leqslant i<j\leqslant r} m_{ij}\mathscr{H}^{N-1}(\partial^* S_i \cap \partial^* S_j \cap \Omega).$$

First, we prove the case $r=2$, and take it as a lemma. Denote $a=a_1$, $b=a_2$, $m=m_{12}$, $E=S_1$.

Lemma 3.3.6. *For any* $u\in \mathrm{BV}(\Omega;\{a,b\})$, *we have*

$$\Gamma-\limsup_{\varepsilon\to 0^+}\mathscr{F}_{\varepsilon}(u)\leqslant m\,\mathrm{Per}_{\Omega}(\{u=a\}).$$

Proof. Suppose $u\in \mathrm{BV}(\Omega;\{a,b\})$, i.e., $u=a\chi_E+b(1-\chi_E)$, where E satisfies $\mathrm{Per}_{\Omega}(E)=|D\chi_E|(\Omega)<+\infty$. First we consider the case $E=\tilde{E}\cap\Omega$, where $\tilde{E}$ is a smooth set in $\mathbb{R}^N$.

Denote $\partial\tilde{E}=M$. Since M is a smooth surface, there exists $\delta_0>0$, such that for any $\delta<\delta_0$, there exists a smooth projection from $U_\delta:=\{x\in\mathbb{R}^N:\mathrm{dist}(x,M)<\delta\}$ to M. Let $\varepsilon_n\to 0$, take $v_n\in W^{k,2}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^d)$ satisfying

$$v_n(t)=\begin{cases} a, & t<-\delta_n/\varepsilon_n,\\ b, & t\geqslant \delta_n/\varepsilon_n\end{cases}$$

and $\lim_{n\to\infty}\int_{\mathbb{R}}(W(v_n)+|v_n^{(k)}|^2)\,\mathrm{d}t=m$. Define as follows $u_n:\Omega\to\mathbb{R}^d$:

$$u_n(x):=\begin{cases} v_n(\frac{\tilde{d}_M(x)}{\varepsilon_n}), & x\in U_n\cap\Omega,\\ a, & x\in E\setminus U_n,\\ b, & x\in\Omega\setminus(E\cup U_n),\end{cases}$$

where $\tilde{d}_M:\mathbb{R}^N\to\mathbb{R}$ is the signed distance function of M. $\tilde{d}_M$ is less than 0 in $\tilde{E}$, and $U_n:=U_{\delta_n}$. We select δ_n by the following condition:

$$\delta_n\to 0,\quad \delta_n/\varepsilon_n\to 0.$$

It is easy to see that the above u_n are in $W^{k,2}(\Omega)$. So we have

$$\begin{aligned}\limsup_{n\to\infty}\tilde{\mathscr{F}}_{\varepsilon_n}(u_n)&=\limsup_{n\to\infty}\int_\Omega\left(\frac{W(u_n)}{\varepsilon_n}+\varepsilon_n^{2k-1}|D^k u_n|^2\right)\mathrm{d}x\\ &=\limsup_{n\to\infty}\left\{\int_{U_n}\frac{W(v_n(\tilde{d}_M(x)/\varepsilon_n))}{\varepsilon_n}\,\mathrm{d}x\right.\\ &\qquad\left.+\int_{U_n}\varepsilon_n^{2k-1}\left|v_n^{(k)}\nabla\tilde{d}_M\cdots\nabla\tilde{d}_M/\varepsilon^k+\sum_{i=1}^{k-1}\frac{1}{\varepsilon^i}v_n^{(i)}P_i\right|^2\mathrm{d}x\right\},\end{aligned}$$

where P_i is the sum of the multiplication of some derivatives of $\tilde{d}_M$ of order less than k.

Now we take the following variable transform: $x:=F(y,t)$, where $F:M\times(-\delta_0/2,\delta_0/2)\to U_{\delta_0/2}$ is a differential homotopy, $F(y,t):=y+t\nu(y)$, $\nu(y)$ is the outer normal with respect to $\tilde{E}$ at y of M. We denote the Jacobi of $F(y,t)$ by $J(y,t)$.

Now

$$\begin{aligned}
&\limsup \mathscr{F}_{\varepsilon_n}(u_n)\\
&\quad\leqslant \limsup_{n\to\infty}\left\{\int_M\int_{-\delta_n}^{\delta_n}\left(\frac{W(v_n(t/\varepsilon_n))}{\varepsilon_n}+\varepsilon_n^{2k-1}\frac{|v_n^{(k)}(t/\varepsilon_n)|^2}{\varepsilon_n^{2k}}\right)J(y,t)\,\mathrm{d}t\,\mathrm{d}\mathscr{H}^{N-1}\right\}\\
&\qquad + C\limsup_{n\to\infty}\int_{-\delta_n}^{\delta_n}\varepsilon_n^{2k-1}\sum_{\substack{i+j<2k\\1\leqslant i\leqslant k\\1\leqslant j\leqslant k}}|v_n^{(i)|(t/\varepsilon_n)}||v_n^{(j)}(t/\varepsilon_n)|\,\frac{1}{\varepsilon_n^{i+j}}\,\mathrm{d}t\,\mathrm{d}\mathscr{H}^{N-1}(y)\\
&\quad=\limsup_{n\to\infty} I_1^{(n)}+\limsup_{n\to\infty} I_2^{(n)}.
\end{aligned}$$

Here we have used the fact that the norm of the gradient of the distance function is 1, and its higher order derivatives are bounded due to the compactness of M. Now we estimate $I_1^{(n)}$ and $I_2^{(n)}$, respectively,

$$\begin{aligned}
I_1^{(n)}(u) &= \int_M\int_{-\delta_n}^{\delta_n}\left(\frac{W(v_n(t/\varepsilon_n))}{\varepsilon_n}+\frac{|v_n^{(k)}(t/\varepsilon_n)|^2}{\varepsilon_n}\right)J(y,t)\,\mathrm{d}t\,\mathrm{d}\mathscr{H}^{N-1}(y)\\
&= \int_M\int_{-\delta_n/\varepsilon_n}^{\delta_n/\varepsilon_n}(W(v_n(s))+|v_n^{(k)}(s)|^2)J(y,s\varepsilon_n)\,\mathrm{d}s\,\mathrm{d}\mathscr{H}^{N-1}(y)\\
&\leqslant \left(\sup_{y\in M,t\in(-\delta_n,\delta_n)} J(y,t)\right)\int_M\int_{\mathbb{R}}(W(v_n(s))+|v_n^{(k)}(s)|^2)\,\mathrm{d}s\,\mathrm{d}\mathscr{H}^{N-1}(y).
\end{aligned}$$

Note that since M is compact, we have that $J(y,t)$ converges uniformly to 1 when $t\to 0$, so we can easily know that

$$\limsup I_1^{(n)}\leqslant m\,\mathrm{Per}_\Omega(E).$$

Next let us prove that $I_2^{(n)}\to 0$. We only need to prove that any term in $I_2^{(n)}$

$$\int_{-\delta_n}^{\delta_n}\varepsilon_n^{2k-1-i-j}|v_n^{(i)}(t/\varepsilon_n)||v_n^{(j)}(t/\varepsilon_n)|\,\mathrm{d}t$$

tends to zero when n tends to infinity.

$$\begin{aligned}
\text{The above} &= \int_{\mathbb{R}}\varepsilon_n^{2k-i-j}|v_n^{(i)}(s)||v_n^{(j)}(s)|\,\mathrm{d}t\\
&\leqslant \varepsilon_n^{2k-i-j}\|v_n^{(i)}(s)\|_{L^2}\|v_n^{(j)}(s)\|_{L^2}.
\end{aligned}$$

Now let $w_n(t):=v_n(t/\varepsilon_n)\in W^{k,2}(-\delta_n,\delta_n)$, then

$$\limsup_{n\to\infty}\int_{-1}^{1}\left(\frac{1}{\varepsilon_n}W(w_n)+\varepsilon_n^{2k-1}|w_n^{(k)}|^2\right)\mathrm{d}t$$

$$=\limsup_{n\to\infty}\int_{-\delta_n}^{\delta_n}\left(\frac{1}{\varepsilon_n}W(w_n)+\varepsilon_n^{2k-1}|w_n^{(k)}|^2\right)\mathrm{d}x$$

$$=\lim_{n\to\infty}\int_{\mathbb{R}}(W(v_n)+|v_n^{(k)}|^2)\,\mathrm{d}t=m.$$

Here we use Theorem 3.1.3 to get

$$\|w_n^{(i)}\|_{L^p(-1,1)}\leqslant C\|w_n^{(k)}\|_{L^2}^{i/k}\|w_n\|_{L^1}^{1-i/k}$$

$$\leqslant C\varepsilon_n^{-((2k-1)/2k)i}\varepsilon_n^{1-i/k},$$

where $p=2k/(2k-i)$. We have $\|w_n^{(k)}\|_{L^p}\leqslant C\varepsilon_n^{-(2k-1)/2}o(1)$. Using the Sobolev Embedding $W^{1,p}(-1,1)\hookrightarrow L^\infty(-1,1)$, we have

$$\|w_n^{(i)}\|_{L^2(-1,1)}<C\varepsilon_n^{-(2k-1)/2}o(1),$$

so that

$$\varepsilon_n^{k-i}\left(\int_{\mathbb{R}}|v_n^{(i)}|^2\right)^{1/2}$$

$$=\varepsilon_n^{k-i}\varepsilon_n^{i-1/2}\left(\int_{-\delta_n}^{\delta_n}|w_n^{(i)}|^2\right)^{1/2}$$

$$=o(1)\to 0.$$

Thus the proof for the case when E is smooth is complete.

Next, let us prove the case when E is not smooth. From Theorem 2.6, there exist a sequence of smooth subsets $E_i=\tilde{E}_i\cap\Omega$, where $\tilde{E}_i$ are bounded smooth sets in $\mathbb{R}^N$, satisfying

$$\chi_{E_i}\to\chi_E \text{ in } L^1(\Omega),\quad |D\chi_{E_i}|(\Omega)\to|D\chi_E|(\Omega).$$

We can construct u_i^n for each E_i, then by using a diagonal argument, the proof for non-smooth case is also complete. □

Now we prove the theorem.

Proof. First we prove the case $N=2$. The following result is given in [4].

Lemma 3.3.7. *Suppose $u(x)=\sum_{i=1}^{k}a_i\chi_{S_i}(x)$, and $S_i\subset\Omega$ does not intersect each other, with finite perimeter, and $|\Omega\setminus\cup_{i=1}^{k}S_i|=0$. Then there exist a sequence of division of*

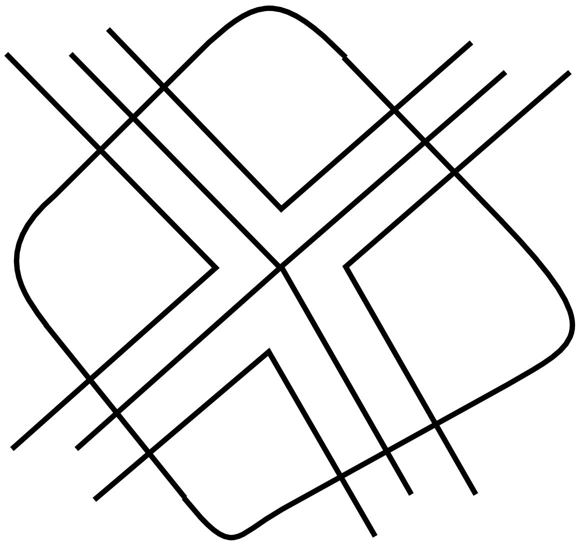

Fig. 2. An illustration of M_δ.

Ω:$\{S_1^n,\ldots,S_k^n\}$, satisfying

(i) *S_i^n is polygonal regions, and $\mathscr{H}^{N-1}(\partial S_i^n \cap \partial\Omega) = 0$ for any $i = 1,\ldots,k$.*
(ii) *Let $u_n(x) = \sum_{i=1}^k a_i \chi_{S_i^k}(x)$, then $u_n \to u$ in $L^1(\Omega)$.*
(iii) *$\int_\Omega u_n(x)\,\mathrm{d}x = \int_\Omega u(x)\,\mathrm{d}x = m$ for any $n \in \mathbb{N}$.*
(iv) *$\lim_{n\to+\infty} \bigvee_{i=1}^k \int_\Omega |D(\varphi_i \circ u_n)| = \bigvee_{i=1}^k \int_\Omega |D(\varphi_i \circ u)|$.*

From this we can easily construct a family of divisions of Ω: $\{E_1^n,\ldots,E_k^n\}$, satisfying

(i') For any $1 \leqslant i < j \leqslant k$, $L_{ij} := \partial E_i^n \cap \partial E_j^n$ is a smooth curve, and at the two ends L_{ij} are straight and conditions (ii)–(iv) in the above lemma.

It is easy to see that if we prove the theorem for $\{E_1,\ldots,E_n\}$ satisfying (i'), then by using a diagonal argument we can prove the general case. So now we just prove the case which satisfies (i').

Let $x \in \Omega$ be the intersection point of two different L_{ij} and suppose further that the curves with one end at x are $\{L_{i_1j_1},\ldots,L_{i_mj_m}\}$. Let

$$U_\delta^l = \{y:\ \mathrm{dist}(y, L_{i_lj_l}) < \delta\},$$

where dist is the unsigned distance function. Let d_δ be the maximum of all $\mathrm{dist}(y_i, x)$, where $\{y_i\}$ are the set of all corner points of $\bigcup_{l=1}^m U_\delta^l$, which is a star shaped region.

It is easy to see that there exists δ_0 sufficiently small, such that we can construct a smooth neighborhood M_{δ_0} of x, such that (as illustrated in Fig. 2):

1. $\mathrm{dist}(x, \partial M_{\delta_0}) > 2d_{\delta_0}$.
2. $\partial M_{\delta_0} \cap U_{\delta_0}^l$ is a straight line for any l.

By noting that the number of x of the above type is finite, we can take a uniform δ_0 to satisfy the above conditions.

For other $\delta \in \mathbb{R}^+$, we define M_δ by the following scaling:

$$M_\delta = \frac{\delta}{\delta_0}(M_{\delta_0} - x) + x.$$

Now we can see that, after cut off a neighborhood described above at each intersection point, we can do the same construction as in the $r=2$ case. So we just need to prove that, after extending u_n to M_{δ_n}, we have

$$\lim_{n\to\infty} \int_{M_{\delta_n}} \left(\frac{W(u_n)}{\varepsilon_n} + \varepsilon_n^{2k-1} |D^k u_n|^2 \right) \mathrm{d}x = 0.$$

Now we extend it. Take any M_{δ_n}. Since outside it we have already constructed u_n, we just need to construct $u_n \in W^{k,2}(M_{\delta_n})$ under the following conditions:

$$u_n|_{\partial M_{\delta_n}} \in W^{k,2}(\partial M_{\delta_n}) \text{ takes known value, } \frac{\partial^l u_n}{\partial v^l} = 0,$$

where $l = 1, \ldots, k-1$, v is the outer normal vector of M_{δ_n}, and $\|u_n\|_{L^\infty}$ is uniformly bounded in n.

From the construction method of M_δ, we can just solve this problem for M_1, then transform it to M_{δ_n} by scaling. Now we begin the construction in M_1. Denote by $\tilde{v}_n : 9M_1 \to \mathbb{R}^d$ the boundary value, which is known, after scaling to ∂M_1.

Since M_1 is a smooth open set, there exists $R > 0$, such that $B_{x,2R} \subset\subset M_1$. Define $U := \bar{M}_1 \setminus (B_{x,R} \cup L)$, $V := (0,1) \times [0,1]$, where L is a line connecting $\partial B_{x,R}$ and ∂M_1. It is easy to know that there exists a one-to-one mapping $\varphi : U \cup L \to V$, such that $\varphi \in C^\infty$, $\varphi^{-1} \in C^\infty$, which maps $\partial B_{x,R} \setminus L$ and $\partial M_1 \setminus L$ to $(0,1) \times \{0\}$ and $(0,1) \times \{1\}$, respectively, and at the same time maps the normal on L to a direction parallel to the direction $(1,0)$, such that the derivatives of any order of φ and φ^{-1} are bounded.

Define $f(x) : [0,1] \to \mathbb{R}^d$ as $f(x) := \tilde{v} \circ \varphi^{-1}(x,1)$. Note that f can be extended to $\mathbb{R}$ with period 1, and $f \in W^{k,2}[0,1]$.

Next we take $g(y) : [0,1] \to \mathbb{R}$ to be C^∞, satisfying

$$g(0) = 0, \qquad g(1) = 1,$$

$$g^{(l)}(0) = g^{(l)}(1) = 0, \quad \forall l \in \mathbb{N}.$$

Obviously such a function exists.

Now we construct $\tilde{u} : M_1 \to \mathbb{R}^d$ as follows:

$$\tilde{u}(x) = \begin{cases} (g \cdot f) \circ \varphi(x), & x \in M_1 \setminus B_{x,R}, \\ \text{extend by continuity}, & x \in L, \\ 0, & x \in B_{x,R}. \end{cases}$$

It is easy to see

$$\|\tilde{u}_n\|_{L^\infty(M_1)} \leqslant \|\tilde{v}_n\|_{L^\infty(\partial M_1)},$$

$$\|\tilde{u}_n\|_{W^{k,2}(M_1)} \leqslant C(\varphi, g) \|\tilde{v}_n\|_{W^{k,2}(\partial M_1)},$$

i.e., such $\tilde{u}_n$ after extension satisfies our conditions.

Using scaling, we have

$$\int_{M_{\delta_n}} \left(\frac{W(u_n)}{\varepsilon_n} + \varepsilon_n^{2k-1} |D^k u_n|^2 \right) \mathrm{d}x$$

$$= \int_{M_1} \left(\frac{W(\tilde{u}_n)}{\varepsilon_n} + \varepsilon_n^{2k-1} |D^k \tilde{u}_n\|^2 \frac{1}{\delta_n^{2k}} \right) \delta_n^2 \,\mathrm{d}x$$

$$= \int_{M_1} \left(\frac{\delta_n^2}{\varepsilon_n} W(\tilde{u}_n) + \frac{\varepsilon^{2k-1}}{\delta^{2k-2}} |D^k \tilde{u}_n|^2 \right) \mathrm{d}x.$$

Thus, by taking $\delta_n = o(\varepsilon_n^{1/2})$, we have proved that the first term tends to zero.

Now we estimate the second term:

$$\int_{M_1} |D^k \tilde{u}_n|^2 \,\mathrm{d}x \leqslant C \int_{\partial M_1} |\tilde{v}_n^{(k)}|^2 \,\mathrm{d}s$$

$$= C \sum_l \int_{\partial M_1 \cap U_1^l} |\tilde{v}_n^{(k)}|^2 \,\mathrm{d}s$$

$$\leqslant C \sum_l \int_{\mathbb{R}} \frac{\delta_n^{2k}}{\varepsilon_n^{2k}} |v_{ij}^{n^{(k)}}|^2 \frac{\varepsilon_n}{\delta_n} \,\mathrm{d}s'$$

$$\leqslant C \frac{\delta_n^{2k-1}}{\varepsilon_n^{2k-1}},$$

where v_{ij}^n is the v^n in the $r = 2$ case, just replacing a, b with a_i, a_j.

Thus it is easy to see that the second term tends to zero too. This ends the proof.

Note that in [4], Lemma 3.3.7 here applies to any $\Omega \in \mathbb{R}^N$, so we can prove the same result similarly in cases $N \geqslant 3$. $\square$

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