Optimal Design of Noise-Enhanced Binary Threshold Detector Under AUC Measure

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***Abstract*—This letter considers the binary threshold system (TS) based detector for a general binary testing problem. First, the optimal binary TS that maximizes the area under the ROC curve (AUC), where ROC stands for the receiver operating characteristic, is derived. Then the noise-enhanced effect is investigated. The optimal noise that can achieve the maximum AUC is derived and shown to be deterministic. An example is shown to help justify the derived results.**

***Index Terms*—AUC, noise-enhanced effect, threshold detector, threshold system.**

I. Introduction

FOR a binary detection problem, the optimal likelihood ratio (LR) test is not always possible due to unknown and changing parameters, lack of robustness, and high complexity. It is therefore necessary to resort to simpler suboptimal detectors. Threshold system (TS) based detector, or threshold detector (TD), is one such detector. It has been shown to achieve good detectability and high robustness in addition to its simplicity in implementation for problems with non-Gaussian noise [1]–[3]. For DC signal detection with known noise, a maximum a-posteriori probability detector for a given binary TS was proposed in [1]. But the optimal TS design was not addressed. We filled this gap in [2] by finding the optimal TS under Neyman-Pearson (NP) criterion. In [3], for an arbitrary known signal detection in non-Gaussian noise with unknown probability density function (pdf), we proposed an optimal TD and analyzed its properties.

In this letter, we derive the optimal binary TS that maximizes the AUC, defined as the area under receiver operating characteristic (ROC) curve. We also investigate the noise-enhanced effect. The idea is to inject additional independent "noise" into the observation for better performance [4]–[9]. It is shown in [4] that when the binary TS in a TD is not optimal, for a given probability of false alarm ($P_{\rm FA}$), the probability of detection (P_D) can be increased by adding independent white Gaussian noise (WGN). Under Bayesian criterion, it has been shown in [5], [9] that the optimal noise is deterministic. Under the NP criterion, the noise-enhanced effect is investigated and the optimal noise pdf is derived in [6]–[8]. For a given $P_{\rm FA}$, by using results in [6]–[8], the optimal noise pdf that leads to the highest P_D can be obtained. But the derived optimal noise pdf is implicitly represented and in general difficult to obtain in closed-form. A numerical method for finding the optimal noise pdf was proposed in [8]. With the NP criterion, the optimal noise pdf depends on the desired level of $P_{\rm FA}$. This further increases computational cost. In this letter, for a general binary detection problem with binary TD, we derive the best noise-enhanced effect under the AUC measure. The optimal noise pdf is derived in a simple form. The computational cost in finding the optimal noise is very low. An illustrative example is presented as well.

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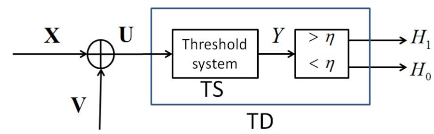

Fig. 1. Noise-enhanced binary TS based detector.

II. Problem Statement and AUC Measure

We consider the following general binary detection problem:

$$\begin{cases} H_0: & \mathbf{X} \sim f_{\mathbf{X}}(\mathbf{x}; H_0) \triangleq f_0(\mathbf{x}) \\ H_1: & \mathbf{X} \sim f_{\mathbf{X}}(\mathbf{x}; H_1) \triangleq f_1(\mathbf{x}) \end{cases}, \tag{1}$$

where $\mathbf{X} \in \mathbb{R}^N$, $f_{\mathbf{X}}(\mathbf{x}; H_i)$ abbreviated as $f_i(\mathbf{x})$, is the pdf's of $\mathbf{X}$ under hypotheses H_i, for $i = 0$ or 1.

We will use a binary TD with possible noise-enhanced effect. Binary TD, other than its low complexity, has been shown to achieve good detectability and high robustness, especially for noises with a heavy-tail pdf [1]–[3]. The detector structure is shown in Fig. 1. First an N-dimension noise $\mathbf{V}$ is added to the original observation $\mathbf{X}$ to produce the new observation $\mathbf{U}$ for possible noise-enhanced effect. Note that $\mathbf{V}$ should be independent of $\mathbf{X}$. Then $\mathbf{U}$ is applied to the binary TS and converted into a binary signal $Y \in \{0, 1\}$. Finally a decision H_1 or H_0 is made.

Let $f_{\mathbf{V}}(\mathbf{x})$ be the pdf of the additional noise $\mathbf{V}$. Since $\mathbf{U} = \mathbf{X} + \mathbf{V}$, we have

$$f_{\mathbf{U}}(\mathbf{u}; H_0) = f_0(\mathbf{u}) * f_{\mathbf{V}}(\mathbf{u}) = \int_{\mathbb{R}^N} f_0(\mathbf{u}-\mathbf{x}) f_{\mathbf{V}}(\mathbf{x}) d\mathbf{x},$$

$$f_{\mathbf{U}}(\mathbf{u}; H_1) = f_1(\mathbf{u}) * f_{\mathbf{V}}(\mathbf{u}) = \int_{\mathbb{R}^N} f_1(\mathbf{u}-\mathbf{x}) f_{\mathbf{V}}(\mathbf{x}) d\mathbf{x},$$

where $*$ denotes convolution. The function of the binary TS can be expressed as

$$Y = T(\mathbf{u}) = \begin{cases} 1 & \mathbf{u} \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}, \tag{2}$$

where $\mathcal{A}$ is a subset in $\mathbb{R}^N$. The binary TS model is also general, including TS's in [2]–[6] as 1-dimensional special cases.

The decision-making is given by $T(\mathbf{u}) \underset{H_0}{\overset{H_1}{\gtrless}} \eta$, which can be completely characterized by the critical function [6]:

$$\phi(\mathbf{u},\eta)=\begin{cases}1: & T(\mathbf{u})>\eta\\ \nu: & T(\mathbf{u})=\eta\\ 0: & T(\mathbf{u})<\eta\end{cases}, \tag{3}$$

where $\nu\in[0,1]$. For $i=0,1$ define

$$F_i(\mathbf{x},\eta)\triangleq\int_{\mathbb{R}^N}\phi(\mathbf{u},\eta)f_i(\mathbf{u}-\mathbf{x})d\mathbf{u}. \tag{4}$$

We have

$$\begin{aligned}P_D(\eta)&=\int_{\mathbb{R}^N}\phi(\mathbf{u},\eta)f_{\mathbf{U}}(\mathbf{u};H_1)d\mathbf{u}\\ &=\int_{\mathbb{R}^N}f_{\mathbf{V}}(\mathbf{x})F_1(\mathbf{x},\eta)d\mathbf{x}, \qquad (5)\\ P_{\mathrm{FA}}(\eta)&=\int_{\mathbb{R}^N}\phi(\mathbf{u},\eta)f_{\mathbf{U}}(\mathbf{u};H_0)d\mathbf{u}\\ &=\int_{\mathbb{R}^N}f_{\mathbf{V}}(\mathbf{x})F_0(\mathbf{x},\eta)d\mathbf{x}. \qquad (6)\end{aligned}$$

Note that if $\mathbf{V}=\mathbf{0}$ (no added noise), we have $\mathbf{U}=\mathbf{X}$, thus

$$P_D(\eta)=F_1(\mathbf{0},\eta), P_{\mathrm{FA}}(\eta)=F_0(\mathbf{0},\eta). \tag{7}$$

One widely used criterion in signal detection is NP criterion. With NP criterion, the optimal TS design depends on the given level of P_{FA}. In general, the optimization of $T(\mathbf{u})$ is computationally costly and sensitive to P_{FA} value [2], [10]. Similar problems exist in finding the best noise pdf for the noise-enhanced effect.

In this letter, we use AUC as the performance measure, which is defined as the area enclosed by the curve (P_{FA}, P_D) together with the lines $P_D=0$ and $P_{\mathrm{FA}}=1$. It is shown to be a valid measure of detection capacity [11], [12]. With the AUC measure, the optimal TS and the optimal noise pdf are independent of the P_{FA} level, thus can lead to low computational load and practical and robust designs.

When both $P_D(\eta)$ and $P_{\mathrm{FA}}(\eta)$ are continuous piecewise differentiable functions, AUC can be calculated through either

$$\mathrm{AUC}=-\int_{-\infty}^{\infty}P_D(\eta)\frac{\partial P_{\mathrm{FA}}(\eta)}{\partial\eta}d\eta \tag{8}$$

or

$$\mathrm{AUC}=\frac{1}{2}+\frac{1}{2}\int_{-\infty}^{\infty}\left(P_{\mathrm{FA}}(\eta)\frac{\partial P_D(\eta)}{\partial\eta}-P_D(\eta)\frac{\partial P_{\mathrm{FA}}(\eta)}{\partial\eta}\right)d\eta. \tag{9}$$

The latter follows from Green's formula. When the functions involved are not continuous, however, the two formulas are not equivalent anymore. In such cases, (8) often leads to incorrect answers. Nevertheless, one can verify through smoothing that (9) still gives the correct answer even for discontinuous $P_D(\eta), P_{\mathrm{FA}}(\eta)$, due to the total cancellation of the ill-defined terms [13]. In the following we will use (9) whenever analytic calculation of AUC is needed.

In this letter, for the detection problem in (1), using the TD in Fig. 1, we solve the following two problems: 1) find the AUC-maximizing binary TS, when there is no added noise enhancement; and 2) find the AUC-maximizing noise pdf, for a given TS.

III. AUC-Maximizing Binary TS

Theorem 1: When no noise is added, the AUC-maximizing binary TS is

$$Y=T(\mathbf{x})=\begin{cases}1 & \mathbf{x}\in\mathcal{A}_{\mathrm{opt}}=\{\mathbf{x}:f_1(\mathbf{x})\geq f_0(\mathbf{x})\}\\ 0 & \text{elsewhere}\end{cases}. \tag{10}$$

Proof: From (2) and (3), we have

$$\phi(\mathbf{u},\eta)=\begin{cases}0 & \eta>1 \text{ or } (0<\eta\leq 1,\mathbf{u}\notin\mathcal{A})\\ \nu & (\eta=1,\mathbf{u}\in\mathcal{A}) \text{ or } (\eta=0,\mathbf{u}\notin\mathcal{A})\\ 1 & \eta<0 \text{ or } (0\leq\eta<1,\mathbf{u}\in\mathcal{A})\end{cases}. \tag{11}$$

Define $p_i\triangleq\int_{\mathbf{u}\in\mathcal{A}}f_i(\mathbf{u})d\mathbf{u}, i=0,1$. Using (11) in (7) gives

$$(P_{\mathrm{FA}},P_D)=\begin{cases}(0,0) & \eta>1\\ (\nu p_0,\nu p_1) & \eta=1\\ (p_0,p_1) & 0<\eta<1\\ (\nu+p_0(1-\nu),\nu+p_1(1-\nu)) & \eta=0\\ (1,1) & \eta<0\end{cases}. \tag{12}$$

Since $\nu\in[0,1]$, the ROC of this TD is the combination of the segment from $(0,0)$ to (p_0,p_1) and the segment from (p_0,p_1) to $(1,1)$. The AUC is thus the area of the triangle $\triangle(0,0)(p_0,p_1)(1,1)$ plus 1/2, which is

$$\mathrm{AUC}=\frac{1}{2}+\frac{1}{2}(p_1-p_0)=\frac{1}{2}+\frac{1}{2}\int_{\mathcal{A}}[f_1(\mathbf{x})-f_0(\mathbf{x})]d\mathbf{x}.$$

The AUC-maximizing $\mathcal{A}_{\mathrm{opt}}$ and TS are thus as in (10). ■

Remark 1: We can obtain the same AUC using (9). Formula (8), however, does not produce the correct result.

IV. Optimal Noise-Enhanced Effect

In this section, we derive the optimal noise pdf $f_{\mathbf{V}}(\mathbf{v})$ that maximizes the AUC for a given binary TS. Define

$$H(\mathbf{x},\mathbf{y})\triangleq\int_{\mathbb{R}}F_0(\mathbf{x},\eta)\frac{\partial F_1(\mathbf{y},\eta)}{\partial\eta}d\eta-\int_{\mathbb{R}}F_1(\mathbf{x},\eta)\frac{\partial F_0(\mathbf{y},\eta)}{\partial\eta}d\eta. \tag{13}$$

Using (5) and (6) in (9), we have the AUC calculation in (14):

$$\begin{aligned}\mathrm{AUC}&=\frac{1}{2}+\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}^N}f_{\mathbf{V}}(\mathbf{x})F_0(\mathbf{x},\eta)d\mathbf{x}\right)\times\left(\int_{\mathbb{R}^N}f_{\mathbf{V}}(\mathbf{y})\frac{\partial F_1(\mathbf{y},\eta)}{\partial\eta}d\mathbf{y}\right)d\eta\\ &\quad-\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}^N}f_{\mathbf{V}}(\mathbf{x})F_1(\mathbf{x},\eta)d\mathbf{x}\right)\times\left(\int_{\mathbb{R}^N}f_{\mathbf{V}}(\mathbf{y})\frac{\partial F_0(\mathbf{y},\eta)}{\partial\eta}d\mathbf{y}\right)d\eta\\ &=\frac{1}{2}+\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}f_{\mathbf{V}}(\mathbf{x})f_{\mathbf{V}}(\mathbf{y})\times\left(\int_{\mathbb{R}}F_0(\mathbf{x},\eta)\frac{\partial F_1(\mathbf{y},\eta)}{\partial\eta}d\eta-\int_{\mathbb{R}}F_1(\mathbf{x},\eta)\frac{\partial F_0(\mathbf{y},\eta)}{\partial\eta}d\eta\right)d\mathbf{x}d\mathbf{y}\\ &=\frac{1}{2}+\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}f_{\mathbf{V}}(\mathbf{x})f_{\mathbf{V}}(\mathbf{y})H(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}.\end{aligned} \tag{14}$$

The noise pdf optimization is thus equivalent to

$$\arg\max_{f_{\mathbf{V}}(\mathbf{x})} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_{\mathbf{V}}(\mathbf{x})f_{\mathbf{V}}(\mathbf{y})H(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y} \quad (15)$$

$$\text{s.t.} \int_{\mathbb{R}^N} f_{\mathbf{V}}(\mathbf{x})d\mathbf{x} = 1, \quad f_{\mathbf{V}}(\mathbf{x}) \geq 0. \quad (16)$$

The conditions in (16) apply since $f_{\mathbf{V}}(\mathbf{x})$ is a pdf.

Theorem 2: For a given binary TS in (2), define

$$G(\mathbf{x}) \triangleq \int_{\mathcal{A}} [f_1(\mathbf{u}-\mathbf{x}) - f_0(\mathbf{u}-\mathbf{x})]d\mathbf{u}. \quad (17)$$

Let $\mathbf{x}_{\text{opt}}$ be the maximum point of $G(\mathbf{x})$, i.e.,

$$\mathbf{x}_{\text{opt}} = \arg\max_{\mathbf{x}} G(\mathbf{x}). \quad (18)$$

The optimal noise pdf that maximizes the AUC is

$$f_{\mathbf{V}_{\text{opt}}}(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_{\text{opt}}), \quad (19)$$

where $\delta(\cdot)$ is the Dirac delta function.

Proof: See the Appendix. ■

This theorem says that the optimal added $\mathbf{V}$ is deterministic, whose value is the $\mathbf{x}_{\text{opt}}$ defined in (18). This is equivalent to conducting an optimal mean shift on the observation. Once the structure of the optimal pdf is found, the determination of the optimal value is straightforward. Note that $\mathbf{x}_{\text{opt}}$ may not be unique because in general there may exist multiple values of $\mathbf{x}$ that give the same maximum $G(\mathbf{x})$. Using any one or any randomization of these $\mathbf{x}_{\text{opt}}$'s will result in the same maximum AUC. When a randomization is used, an alternatively interpretation is that the optimum detection can be achieved by adding a random noise.

From the proof of Theorem 2, for a deterministic added noise $\mathbf{v}$, we have, from (14) and (29), $\text{AUC} = [1 + G(\mathbf{v})]/2$. We can thus determine whether noise-enhanced effect exists by comparing $G(\mathbf{v})$ with $G(\mathbf{0})$.

Corollary 1: (Existence of noise-enhanced effect)

1) For $\mathbf{v} \in \mathbb{R}^n$, if $G(\mathbf{v}) > G(\mathbf{0})$, the AUC of the TD can be improved by adding the constant $\mathbf{v}$.
2) If the TS in the TD is optimal in the AUC sense, the AUC of the TD cannot be increased via adding noise.

Proof: The first part of the corollary can be seen directly from the AUC formula. Now we prove the second part. It is shown in Theorem 2 that the best noise is deterministic. With the optimal TS design in Theorem 1, $G(\mathbf{0})$ is the maximum of $G(\mathbf{v})$, thus AUC can no long be improved by any $\mathbf{v}$. ■

We now discuss how to find $\mathbf{x}_{\text{opt}}$ in (18), the global maximum point of $G(\mathbf{x})$. First, candidate $\mathbf{x}_c$'s should satisfy $G'(\mathbf{x}_c) = 0$ and $G''(\mathbf{x}_c) \leq 0$. Thus we first find $\mathbf{x}_c$'s that satisfy the two conditions, then $\mathbf{x}_{\text{opt}}$ is one of the $\mathbf{x}_c$'s resulting in the largest value of $G(\mathbf{x})$. This can be done efficiently using standard numerical algorithms such as Newton's method.

In [6], [7], under NP criterion, for a given test, the optimal noise pdf was proved to be a combination of at most two delta functions. Although the noise pdf structure was found, the exact optimal pdf is difficult to find. In this work, we adopt the AUC measure and derive the optimal noise pdf in a semi-closed form, the calculation of which is significantly simpler. Under the NP criterion, the optimal noise pdf changes with η and P_{FA}, hence the design can be sensitive to them. AUC leads to an optimal noise independent of η and P_{FA}, and intuitively can be more robust. It is noteworthy that for a given P_{FA} and under perfect design, the proposed scheme may be inferior in P_D to those in [6], [7] since the goal here is to maximize the AUC not the P_D for a particular P_{FA}.

V. An Example

We now use an example to illustrate the results in Sections III and IV. We consider a one-dimensional mean-shift Gaussian mixture detection. Let

$$f_0(x) = \frac{1}{2}\mathcal{N}(x;\mu,\sigma^2) + \frac{1}{2}\mathcal{N}(x;-\mu,\sigma^2),$$
$$f_1(x) = \frac{1}{2}\mathcal{N}(x;\mu+s,\sigma^2) + \frac{1}{2}\mathcal{N}(x;-\mu+s,\sigma^2),$$

where $\mathcal{N}(x;\mu,\sigma^2)$ is Gaussian distribution with mean μ and variance σ^2. We set $\mu = 3, \sigma = 1, s = 0.5$. The optimal binary TS, denoted as TS_{opt}, can be derived using Theorem 1 to be

$$\text{TS}_{\text{opt}} : Y = \begin{cases} 1 & x \in [\tau_1, \tau_2] \cup [\tau_3, \infty) \\ 0 & \text{elsewhere} \end{cases}, \quad (20)$$

where $\tau_1 = -2.75$, $\tau_2 = 0.25$, and $\tau_3 = 3.25$.

First we consider the TD with TS_{opt}. From (20), the resulting $G(x)$ is expressed as (21), where $Q(x) = \int_x^\infty (1)/(\sqrt{2\pi})\exp(-(x^2)/(2))$.

$$G_{\text{opt}}(x) = \frac{1}{2}\sum_{i=1}^{3}(-1)^{i-1} \times [Q(\tau_i - 3.5 - x) - Q(\tau_i - 3 - x) + Q(\tau_i + 2.5 - x) - Q(\tau_i + 3 - x)]. \quad (21)$$

$$-\frac{\partial F_i(\mathbf{x},\eta)}{\partial \eta} = c_i(\mathbf{x})\delta(\eta - 1) + d_i(\mathbf{x})\delta(\eta). \quad (22)$$

$$\begin{aligned} -\int_{\mathbb{R}} F_1(\mathbf{x},\eta)\frac{\partial F_0(\mathbf{y},\eta)}{\partial \eta}d\eta &= F_1(\mathbf{x},1)c_0(\mathbf{y}) + F_1(\mathbf{x},0)d_0(\mathbf{y}) \\ &= c_1(\mathbf{x}) + \nu d_1(\mathbf{x})d_0(\mathbf{y}) - (1-\nu)c_1(\mathbf{x})c_0(\mathbf{y}) \end{aligned} \quad (23)$$

$$\begin{aligned} \int_{\mathbb{R}} F_0(\mathbf{x},\eta)\frac{\partial F_1(\mathbf{y},\eta)}{\partial \eta}d\eta &= -c_0(\mathbf{x}) - \nu d_1(\mathbf{y})d_0(\mathbf{x}) + (1-\nu)c_1(\mathbf{y})c_0(\mathbf{x}) \end{aligned} \quad (24)$$

$$K(\mathbf{x},\mathbf{y}) \triangleq \nu[d_1(\mathbf{x})d_0(\mathbf{y}) - d_1(\mathbf{y})d_0(\mathbf{x})] + (1-\nu)[c_1(\mathbf{y})c_0(\mathbf{x}) - c_1(\mathbf{x})c_0(\mathbf{y})] \quad (25)$$

After numerical calculation, it follows that $x_{\text{opt}} = 0$. This justifies the second part of Corollary 1 that noise-enhanced effect cannot occur if the TD is optimal.

Next, we consider the TD with a suboptimal TS, denoted as TS_{nopt}, given by

$$Y = \begin{cases} 1 & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}. \quad (26)$$

Similarly, we can obtain the $G(x)$ as in (27), which leads to two optimal solutions: $x_{\text{opt}} = -3.25$ or $x_{\text{opt}} = 2.75$. They both result in a higher $G = 0.0988$ than $G(0) = 0.0024$. Based on the first part of Corollary 1, the noise-enhanced effect appears:

$$G(x) = \frac{1}{2}[Q(-3.5 - x) + Q(2.5 - x) - Q(-3 - x) - Q(3 - x)] \quad (27)$$

We compare the AUCs of the five cases: 1) using TS_{opt}, 2) using TS_{nopt}, 3) using TS_{nopt} and adding $\mathbf{V}_{\text{opt}} = -3.25$,

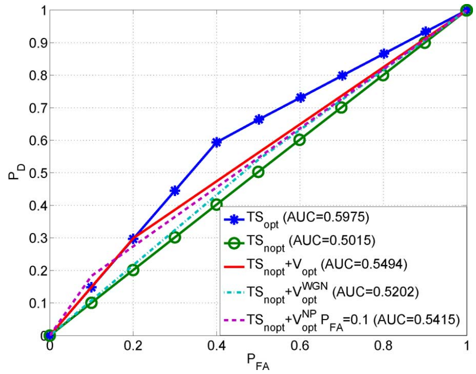

Fig. 2. ROCs of different detector designs.

4) using TS_{nopt} and adding the optimal zero-mean WGN: $f_{\mathbf{V}}(x) = \mathcal{N}(x;0,\sigma^2)$, and 5) using TS_{nopt} and adding the optimal noise under NP criterion for $P_{\mathrm{FA}} = 0.1$. For Case 4, with added WGN, we have

$$f_0(x) = \frac{1}{2}\mathcal{N}(x;3,1+\sigma^2) + \frac{1}{2}\mathcal{N}(x;-3,1+\sigma^2),$$
$$f_1(x) = \frac{1}{2}\mathcal{N}(x;3.5,1+\sigma^2) + \frac{1}{2}\mathcal{N}(x;-2.5,1+\sigma^2).$$

The optimal σ can be shown to be:

$$\sigma_{\mathrm{opt}} = \arg\max_{\sigma} \int_0^{\infty} [f_1(x) - f_0(x)]dx = 2.81.$$

For Case 5, we have $f_{V_{\mathrm{opt}}}(v) = \delta(v+3.86)$.

The ROC and AUC results are shown in Fig. 2. We can see that with $\mathrm{TS}_{\mathrm{opt}}$, the AUC is 0.5975 which is larger than that with TS_{nopt}, which is 0.5015. This justifies Theorem 1. When TS_{nopt} is used, noise-enhanced effect happens by adding V_{opt}. The AUC of using TS_{nopt} and V_{opt} is 0.5494, which is lower than that of $\mathrm{TS}_{\mathrm{opt}}$. This is because the structure of TS_{nopt} is not optimal. With TS_{nopt}, adding the best deterministic V_{opt} is better than adding the best WGN, whose AUC is 0.5202. This justifies Theorem 2. Finally, under NP criterion [6], $V_{\mathrm{opt}}^{\mathrm{NP}}$ depends on the P_{FA} value. The achieved P_D using $V_{\mathrm{opt}}^{\mathrm{NP}}$ is higher than that using V_{opt} at the specific P_{FA} value. However, the latter achieves a higher AUC.

VI. Conclusion

We investigated the general binary detection problem using a binary threshold system based detector. The AUC is adopted as the performance measure for its simplicity and robustness. First the AUC-maximizing TS was derived. We then considered noise-enhanced effect and showed that the AUC-maximizing noise is deterministic. Performance of the proposed design was shown via an example and compared with other designs.

Appendix
Proof of Theorem 2

For $i = 0,1$, define

$$c_i(\mathbf{x}) \triangleq \int_{\mathbf{u}\in\mathcal{A}} f_i(\mathbf{u}-\mathbf{x})d\mathbf{u},\ d_i(\mathbf{x}) \triangleq \int_{\mathbf{u}\notin\mathcal{A}} f_i(\mathbf{u}-\mathbf{x})d\mathbf{u}.$$

We have $c_0(\mathbf{x}) + d_0(\mathbf{x}) = c_1(\mathbf{x}) + d_1(\mathbf{x}) = 1$. Using (11) in (4) gives

$$F_i(\mathbf{x},\eta) = \begin{cases} 0 & \eta > 1 \\ \nu c_i(\mathbf{x}) & \eta = 1 \\ c_i(\mathbf{x}) & 0 < \eta < 1 \\ c_i(\mathbf{x}) + \nu d_i(\mathbf{x}) & \eta = 0 \\ 1 & \eta < 0 \end{cases}.$$

We then have the results in (22)–(24).

Now using (23) and (24) in (13), we have

$$H(\mathbf{x},\mathbf{y}) = \int_{\mathbf{u}\in\mathcal{A}} [f_1(\mathbf{u}-\mathbf{x}) - f_0(\mathbf{u}-\mathbf{x})]d\mathbf{u} + K(\mathbf{x},\mathbf{y}), \quad (28)$$

where $K(\mathbf{x},\mathbf{y})$ is defined in (25). It can be shown straightforwardly that $K(\mathbf{x},\mathbf{y})$ is skew-symmetric, i.e., $K(\mathbf{x},\mathbf{y}) = -K(\mathbf{y},\mathbf{x})$. Because the integral of the skew-symmetric terms is zero, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_{\mathbf{V}}(\mathbf{x})f_{\mathbf{V}}(\mathbf{y})H(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y} = \int_{\mathbb{R}^N} f_{\mathbf{V}}(\mathbf{x})G(\mathbf{x})d\mathbf{x}, \quad (29)$$

with $G(\mathbf{x})$ defined in (17). By using Holder's inequality,

$$\int_{\mathbb{R}^N} f_{\mathbf{V}}(\mathbf{x})G(\mathbf{x})d\mathbf{x} \le \|f_{\mathbf{V}}(\mathbf{x})\|_1 \|G(\mathbf{x})\|_{\infty} = \max G(\mathbf{x})$$

with equality when $f_{\mathbf{V}_{\mathrm{opt}}}(\mathbf{x}) = \delta(\mathbf{x}-\mathbf{x}_{\mathrm{opt}})$ with $\mathbf{x}_{\mathrm{opt}}$ in (18).

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