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Fractional Gagliardo–Nirenberg and Hardy inequalities under Lorentz norms

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ABSTRACT

In this paper, we establish the Gagliardo–Nirenberg inequality under Lorentz norms for fractional Laplacian. Based on special cases of this inequality under Lebesgue norms, we prove the L^p-logarithmic Gagliardo–Nirenberg and Sobolev inequalities. Motivated by the L^2-logarithmic Sobolev inequality, we obtain a fractional logarithmic Sobolev trace inequality in terms of the restriction $\tau_k u$ of u from $\mathbb{R}^n$ to $\mathbb{R}^{n-k}$. Finally, we prove the fractional Hardy inequality under Lorentz norms.

1. Introduction

During the last decades, fractional calculus has received a great deal of attention. Many famous mathematicians have been working in this field and a journal entirely dedicated to the fractional calculus appeared several years ago. The fractional powers of the Laplacian $(-\triangle)^{s/2}$ arise in stochastic theory as the operators associated with symmetric s-stable Levy processes. They play a crucial role in the study of many important PDEs [1–11], among them the quasi-geostrophic equation in geophysical fluid dynamics [12], which is crucial to the understanding of atmosphere and ocean, as well as being an enlightening model of 3D Euler equations and Navier–Stokes equations. To study PDEs with fractional Laplacian, it is necessary to establish several functional inequalities with fractional Laplacians. The main concern of this paper is to prove the fractional Gagliardo–Nirenberg and Hardy inequalities in Lorentz spaces.

First, we prove the fractional Gagliardo–Nirenberg inequality under Lorentz norms:

$$\|u\|_{L^{p,q}(\mathbb{R}^n)} \le A\|(-\triangle)^{s/2}u\|^{\frac{\alpha}{q}}_{L^{p_1,q_1}(\mathbb{R}^n)}\|u\|^{\frac{q-\alpha}{q}}_{L^{p_2,q_2}(\mathbb{R}^n)} \tag{1.1}$$

and obtain an upper bound for the constant A. Similar inequalities in other spaces have been studied extensively, see [1] and the references therein.

When working in Lebesgue spaces, similar to (1.1), we get the fractional Gagliardo–Nirenberg inequality:

$$\|u\|_{L^{p}(\mathbb{R}^n)} \le B^{\frac{\alpha}{p}}\|(-\triangle)^{s/2}u\|^{\frac{\alpha}{p}}_{L^{p_1}(\mathbb{R}^n)}\|u\|^{\frac{p-\alpha}{p}}_{L^{p_2}(\mathbb{R}^n)}, \tag{1.2}$$

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from the sharp fractional Sobolev inequality

$$\|u\|_{L^{\frac{pn}{n-ps}}(\mathbb{R}^n)} \le B\|(-\triangle)^{s/2}u\|_{L^p(\mathbb{R}^n)}. \tag{1.3}$$

The special case of (1.2) plays a significant role in a recent work of Frank and Lenzmann [13]. Cotsiolis and Tavoularis in [14,15] have studied (1.3) for $p = 2$.

Then, combining (1.2) and the convexity of $g(p) : p \longrightarrow p\ln(\int_{\mathbb{R}^n}|f|^{1/p}dx)$, we get the fractional logarithmic Gagliardo–Nirenberg inequality:

$$\exp\left(\left(\frac{1}{q}+\frac{s}{n}-\frac{1}{p_1}\right)\int_{\mathbb{R}^n}\frac{|u(x)|^q}{\|u\|^q_{L^q(\mathbb{R}^n)}}\ln\left(\frac{|u(x)|^q}{\|u\|^q_{L^q(\mathbb{R}^n)}}\right)dx\right) \le \frac{B\|(-\triangle)^{s/2}u\|_{L^{p_1}(\mathbb{R}^n)}}{\|u\|_{L^q(\mathbb{R}^n)}}, \tag{1.4}$$

which implies the fractional logarithmic Sobolev inequality:

$$\exp\left(\frac{s}{n}\int_{\mathbb{R}^n}|u(x)|^p\ln|u(x)|^p dx\right) \le B\|(-\triangle)^{s/2}u\|_{L^p(\mathbb{R}^n)}. \tag{1.5}$$

Merker [16,17] has studied (1.4) for $s = 1$ and applied in doubly nonlinear diffusion equations. Cotsiolis and Tavoularis in [18] established (1.5) for $p = 2$. The case $s = 1$ of (1.5) becomes the classical logarithmic Sobolev inequality, see, e.g., [19–21] and the references therein. Beckner and Pearson in [22] proved that the best constant $(\frac{2}{ne\pi})^{1/2}$ of (1.5) for $p = 2$ and $s = 1$ is the asymptotic of the sharp constant of the L^2-Sobolev inequality. So, for general s, we guess that the best constant of (1.5) is $(\frac{2}{ne\pi})^{s/2}$ which is the asymptotic of the sharp constant B of fractional Sobolev inequality (1.3).

When $p = 2$ and $0 < s < 1$, $(\int_{\mathbb{R}^{n+1}_+}|\nabla_{x,t}h(x,t)|^2t^{1-2s}dxdt)^{1/2}$ is equivalent to the right hand side of (1.5) (see e.g. [23,24]). Here $h(x,t)$ is the extension of $u(x)$ defined through the equation

$$\begin{cases} h(x,0) = u(x), \\ \triangle_x h + \dfrac{1-2s}{t}h_t + h_{tt} = 0. \end{cases} \tag{1.6}$$

Thus, (1.5) can also be thought as a logarithmic Sobolev trace inequality. Note that $(\int_{\mathbb{R}^{n+1}_+}|\nabla_{x,t}h(x,t)|^2t^{1-2s}dxdt)^{1/2}$ is not the L^2-norm of the fractional Laplacian of a function on $\mathbb{R}^{n+1}$. We also deduce a logarithmic Sobolev trace inequality for fractional Laplacian

$$\exp\left(\int_{\mathbb{R}^{n-k}}|(\tau_k u)(x)|^2\ln|\tau_k u(x)|dx\right) \le F\|u\|^{\frac{n-k}{4s-2k}}_{\dot{H}^s(\mathbb{R}^n)}. \tag{1.7}$$

Here $\tau_k u$ is the restriction of u from $\mathbb{R}^n$ to the $n-k$ dimensional hyperplane $\mathbb{R}^{n-k}$. Inequality (1.7) generalizes the corresponding inequalities of Park [25], Xiao [24], and Xiao and Zhai [26].

Our self-contained approach also enables us to prove the fractional Hardy inequality in Lorentz spaces:

$$\left\|\frac{u(x)}{|x|^s}\right\|_{L^{p,q}(\mathbb{R}^n)} \le H\|(-\triangle)^{s/2}u(x)\|_{L^{p,q}(\mathbb{R}^n)} \tag{1.8}$$

and to exhibit an upper bound for the constant H. Note that in a recent work, Frank and Sereinger [27] have established a sharp version of the Hardy inequality; however they have replaced the classical norm by another one to which it is equivalent only for $p = 2$. The fractional Hardy inequalities under Lebesgue norms have been extensively studied by Bogdan and Dyda [28], Dyda [29], Frank and Seiringer [30], Loss and Sloane [31], and Ostrovsky and Sirota [32].

Before proceeding to the subsequent sections, let us introduce several necessary notations. For real number s, the operator $(-\triangle)^{s/2}$ is defined through Fourier transformation by

$$\widehat{(-\triangle)^{s/2}f}(\xi) := (2\pi|\xi|)^s\hat{f}(\xi).$$

For $0 < s < n$, $(-\triangle)^{s/2}$ can be viewed as the inverse of the Riesz potential up to a positive constant, that is,

$$f(x) = K_{s,n}\int_{\mathbb{R}^n}\frac{(-\triangle)^{s/2}f(y)}{|y-x|^{n-s}}dy, \quad \forall x \in \mathbb{R}^n \tag{1.9}$$

with

$$K_{s,n} = \frac{\Gamma\left(\frac{n-s}{2}\right)}{\pi^{n/2}2^s\Gamma(s/2)}.$$

The Riesz potential of a function f is defined by the Riesz kernel $I_s = |x|^{s-n}$ as, for $0 < s < n$,

$$I_s * f(x) = \int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-s}}dy \quad \text{for } x \in \mathbb{R}^n.$$

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For a measurable function defined on an open domain $\mathbb{R}^n$, we define the distribution function $m_f(s)$ of f as

$$m_f(s) = |\{x \in \mathbb{R}^n : |f(x)| > s\}|.$$

Here $|A|$ is the Lebesgue measure of the set $A \subset \mathbb{R}^n$. From the distribution function, we define the non-increasing rearrangement of f on $(0, \infty)$ as

$$f^*(t) = \inf\{s > 0 : m_f(s) \le t\}.$$

For $1 \le p \le \infty$, it is well known that $\|f\|_{L^p(\mathbb{R}^n)} = \|f^*\|_{L^p(0,\infty)}$. This leads to a definition of the classical Lorentz spaces:

$$L^{p,q}(\mathbb{R}^n) = \{f : f \text{ is measuable on } \mathbb{R}^n \text{ with } \|f\|_{L^{p,q}(\mathbb{R}^n)} < \infty\}$$

for $1 \le p \le \infty$ and $1 \le q \le \infty$, with $\|f\|_{L^{p,q}(\mathbb{R}^n)}$ defined by

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q(0,\infty)},$$

where the usual modification has to be made if $q = \infty$. Obviously, $L^p(\mathbb{R}^n) = L^{p,p}(\mathbb{R}^n)$ for $1 \le p \le \infty$. For an introduction to Lorentz spaces, see, for example, [33–35].

For $1 \le p < \infty$ and $1 \le q \le \infty$, we define the Lorentz–Sobolev space $\dot{W}^s L^{p,q}(\mathbb{R}^n)$ as the set of functions satisfying $(-\triangle)^{s/2} f \in L^{p,q}(\mathbb{R}^n)$. When $p = q$, it become the homogeneous Sobolev space $\dot{W}^{s,p}(\mathbb{R}^n)$. If furthermore $p = 2$, $\dot{W}^{s,2}(\mathbb{R}^n) := \dot{H}^s(\mathbb{R}^n)$.

2. Fractional Gagliardo–Nirenberg inequalities

In this section, we first prove the fractional Gargliardo–Nirenberg inequality under Lorentz norms. Then, we get fractional logarithmic Gagliardo–Nirenberg and fractional logarithmic Sobolev inequalities. Finally, motivated by a special case of the fractional logarithmic Sobolev inequalities, we establish a fractional logarithmic Sobolev trace inequality in terms of the restriction $\tau_k f$ of f.

2.1. Fractional Gagliardo–Nirenberg inequalities

Theorem 2.1. *Let $1 \le p < \infty$, $1 \le p_2, q, q_1, q_2 < \infty$, $0 < \alpha < q$, $0 < s < n$ and $1 < p_1 < n/s$. Then the inequality*

$$\|u\|_{L^{p,q}(\mathbb{R}^n)} \le A \|(-\triangle)^{s/2} u\|^{\frac{\alpha}{q}}_{L^{p_1,q_1}(\mathbb{R}^n)} \|u\|^{\frac{q-\alpha}{q}}_{L^{p_2,q_2}(\mathbb{R}^n)} \tag{2.1}$$

holds for

$$\frac{\alpha}{q_1} + \frac{q-\alpha}{q_2} = 1,$$

$$\alpha\left(\frac{1}{p_1} - \frac{s}{n}\right) + (q-\alpha)\frac{1}{p_2} = \frac{q}{p}$$

and

$$A \le \left(\frac{3np_1}{n - p_1 s}\left(\frac{\omega_{n-1}}{n}\right)^{\frac{n-s}{n}}\right)^{\frac{\alpha}{q}}.$$

Proof. Applying the Hölder inequality and simple computation yields

$$\begin{aligned}
\|u\|^q_{L^{p,q}(\mathbb{R}^n)} &= \|t^{\frac{1}{p}-\frac{1}{q}} u^*\|^q_{L^q(0,\infty)} \\
&= \int_0^\infty (t^{\frac{1}{p}-\frac{1}{q}} u^*(t))^q dt \\
&= \int_0^\infty (t^{\frac{1}{p}-\frac{1}{q}} u^*(t))^\alpha (t^{\frac{1}{p}-\frac{1}{q}} u^*(t))^{q-\alpha} dt \\
&\le \left(\int_0^\infty (t^{\frac{1}{p}-\frac{1}{q}} u^*(t))^{r\alpha} dt\right)^{1/r} \left(\int_0^\infty (t^{\frac{1}{p}-\frac{1}{q}} u^*(t))^{r'(q-\alpha)} dt\right)^{1/r'} \\
&\le \|t^{\frac{1}{p}-\frac{1}{q}} u^*(t)\|^\alpha_{L^{r\alpha}(0,\infty)} \|t^{\frac{1}{p}-\frac{1}{q}} u^*(t)\|^{q-\alpha}_{L^{r'(q-\alpha)}(0,\infty)} \\
&\le \|t^{\frac{1}{p}-\frac{1}{q}+\frac{1}{r\alpha}-\frac{1}{r\alpha}} u^*(t)\|^\alpha_{L^{r\alpha}(0,\infty)} \|t^{\frac{1}{p}-\frac{1}{q}+\frac{1}{r'(q-\alpha)}-\frac{1}{r'(q-\alpha)}} u^*(t)\|^{q-\alpha}_{L^{r'(q-\alpha)}(0,\infty)} \\
&\le \|u\|^\alpha_{L^{p_3,q_1}(\mathbb{R}^n)} \|u\|^{q-\alpha}_{L^{p_2,q_2}(\mathbb{R}^n)}
\end{aligned}$$

where r, s satisfy

$$1 < r,\ r' < \infty, \quad \frac{1}{r} + \frac{1}{r'} = 1, \qquad r\alpha \geq 1,$$

and

$$\frac{1}{p_3} = \frac{1}{p} - \frac{1}{q} + \frac{1}{r\alpha}, \qquad q_1 = r\alpha,$$
$$\frac{1}{p_2} = \frac{1}{p} - \frac{1}{q} + \frac{1}{r'(q-\alpha)}, \qquad q_2 = r'(q-\alpha).$$

Note that

$$u(x) = K_{s,n} \int_{\mathbb{R}^n} \frac{(-\triangle)^{s/2} u(y)}{|x-y|^{n-s}} dy.$$

Using the convolution inequality in Lorentz spaces (see O'Neil's [34, Theorem 2.6]), we deduce from $|x|^{s-n} \in L^{\frac{n}{n-s},\infty}(\mathbb{R}^n)$ that

$$\|u\|_{L^{p_3,q_1}(\mathbb{R}^n)} \leq c\|(-\triangle)^{s/2} u\|_{L^{p_1,q_1}(\mathbb{R}^n)}$$

for

$$\frac{1}{p_3} = \frac{1}{p_1} - \frac{s}{n}, \quad 0 < s < n/p_1.$$

It follows from [26, Theorem 2.1, Remark 2.4] that the constant c satisfies

$$c \leq \frac{3np_1}{n - p_1 s} \left(\frac{\omega_{n-1}}{n}\right)^{\frac{n-s}{n}}$$

if $1 \leq q < \infty$. Thus, (2.1) holds for $0 < s < n$, $0 < s < n/p_1$,

$$\frac{\alpha}{q_1} + \frac{q-\alpha}{q_2} = 1, \qquad \alpha\left(\frac{1}{p_1} - \frac{s}{n}\right) + (q-\alpha)\frac{1}{p_2} = \frac{q}{p}. \quad \square$$

When working in Lesbesgue spaces, we need the following sharp fractional Sobolev inequality (2.2) which can be deduced from the sharp Hardy–Littlewood Sobolev inequality:

Lemma 2.2 *([36]).* *For $1 < p < \infty$ and $0 < s < n/p$, we have*

$$\|u\|_{L^{\frac{pn}{n-ps}}(\mathbb{R}^n)} \leq B\|(-\triangle)^{s/2} u\|_{L^p(\mathbb{R}^n)} \tag{2.2}$$

with the best constant

$$B = 2^{-s}\pi^{-s/2} \frac{\Gamma((n-s)/2)}{\Gamma((n+s)/2)} \left(\frac{\Gamma(n)}{\Gamma(n/2)}\right)^{s/n}. \tag{2.3}$$

Similar to Theorem 2.1, we can get the following fractional Gagliardo–Nirenberg inequalities from (2.2).

Corollary 2.3. *Let $1 \leq p, p_2 < \infty$, $0 < \alpha < p < \infty$, $0 < s < n$ and $1 < p_1 < n/s$. We have*

$$\|u(x)\|_{L^p(\mathbb{R}^n)} \leq B^{\frac{\alpha}{p}} \|(-\Delta)^{s/2} u(x)\|_{L^{p_1}(\mathbb{R}^n)}^{\frac{\alpha}{p}} \|u(x)\|_{L^{p_2}(\mathbb{R}^n)}^{\frac{p-\alpha}{p}} \tag{2.4}$$

with

$$\alpha\left(\frac{1}{p_1} - \frac{s}{n}\right) + \frac{p-\alpha}{p_2} = 1$$

and B given by (2.3)

Proof. Hölder's inequality implies

$$\begin{aligned} \int_{\mathbb{R}^n} |u(x)|^p dx &= \int_{\mathbb{R}^n} |u(x)|^\alpha |u(x)|^{p-\alpha} dx \\ &\leq \| |u|^\alpha \|_{L^{\bar{p}}} \| |u|^{p-\alpha} \|_{L^{\bar{r}}}, \qquad \frac{1}{\bar{p}} + \frac{1}{\bar{r}} = 1 \\ &= \|u\|_{L^{\alpha\bar{p}}}^\alpha \|u\|_{L^{(p-\alpha)\bar{r}}}^{p-\alpha}. \end{aligned}$$

Let $\alpha\bar{p} := p_0$ and $(p-\alpha)\bar{r} := p_2$. We have

$$\int_{\mathbb{R}^n} |u(x)|^p dx \le \|u\|_{L^{p_0}}^{\alpha} \|u\|_{L^{p_2}}^{p-\alpha},$$

and $\frac{\alpha}{p_0} + \frac{p-\alpha}{p_2} = 1$. Then, (2.2) yields (2.4) with p_1 satisfying $\frac{1}{p_1} + \frac{n-s}{n} = 1 + \frac{1}{p_0}$. □

Remark 2.4. (i) It follows from the special case $p_1 = q_1$ and $p_2 = q_2$ of (2.1) that the inequality

$$\|u\|_{L^{p,q}(\mathbb{R}^n)} \le A\|(-\triangle)^{s/2}u\|_{L^{p_1}(\mathbb{R}^n)}^{\frac{\alpha}{q}} \|u\|_{L^{p_2}(\mathbb{R}^n)}^{\frac{q-\alpha}{q}} \tag{2.5}$$

with $q = p(1-\frac{\alpha s}{n})$. (2.5) can be thought of as a refinement of (2.4) since $L^{p,r_1} \hookrightarrow L^{p,r_2}$ for $r_1 < r_2$.

(ii) When $q = p_1 = \alpha$, (2.5) becomes

$$\|u\|_{L^{\frac{qn}{n-qs},q}(\mathbb{R}^n)} \le A\|(-\triangle)^{s/2}u\|_{L^q(\mathbb{R}^n)}$$

which can also be thought of as a refinement of (2.2) (see, e.g., [26]).

2.2. Logarithmic Gagliardo–Nirenberg inequalities

We prove fractional logarithmic Gagliardo–Nirenberg inequalities which imply the L^p-logarithmic Sobolev inequalities for fractional Laplacian.

Theorem 2.5. *Let $1 < q < \infty$, $0 < s < n$ and $1 < p_1 < n/s$. Then the inequality*

$$\exp\left(\left(\frac{1}{q} + \frac{s}{n} - \frac{1}{p_1}\right)\int_{\mathbb{R}^n} \frac{|u(x)|^q}{\|u\|_{L^q(\mathbb{R}^n)}^q} \ln\left(\frac{|u(x)|^q}{\|u\|_{L^q(\mathbb{R}^n)}^q}\right)dx\right) \le \frac{B\|(-\triangle)^{s/2}u\|_{L^{p_1}(\mathbb{R}^n)}}{\|u\|_{L^q(\mathbb{R}^n)}} \tag{2.6}$$

holds for

$$\frac{1}{q} + \frac{s}{n} - \frac{1}{p_1} > 0$$

and B given by (2.3).

Proof. The convexity of $g(h) : p \longrightarrow h\ln(\int_{\mathbb{R}^n} |u(x)|^{1/h}dx)$ implies

$$g'(h) = \ln\left(\int_{\mathbb{R}^n} |u(x)|^{1/h}dx\right) - \frac{1}{h}\frac{\int_{\mathbb{R}^n} |u(x)|^{1/h}\ln|u(x)|dx}{\int_{\mathbb{R}^n} |u(x)|^{1/h}dx} \ge \frac{g(h_1) - g(h)}{h_1 - h}$$

for $h > h_1 \ge 0$. By taking $h = \frac{1}{q}$ and $h_1 = \frac{1}{p}$, Merker in [17] established the following logarithmic Hölder inequalities:

$$\int_{\mathbb{R}^n} \frac{|u(x)|^q}{\|u\|_{L^q(\mathbb{R}^n)}^q} \ln\left(\frac{|u(x)|^q}{\|u\|_{L^q(\mathbb{R}^n)}^q}\right)dx \le \frac{p}{p-q}\ln\left(\frac{\|u\|_{L^p(\mathbb{R}^n)}^q}{\|u\|_{L^q(\mathbb{R}^n)}^q}\right)$$

for $0 < q < p \le \infty$. We can choose $p = \frac{nq}{n-qs} \in (q, \infty)$ for $p_2 = q$ and α satisfying the condition of Corollary 2.3 and get

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|u(x)|^q}{\|u\|_{L^q(\mathbb{R}^n)}^q} \ln\left(\frac{|u(x)|^q}{\|u\|_{L^q(\mathbb{R}^n)}^q}\right)dx &\le \frac{p}{p-q}\ln\left(\frac{\left(B^{\frac{\alpha}{p}}\|(-\triangle)^{s/2}u\|_{L^{p_1}(\mathbb{R}^n)}^{\frac{\alpha}{p}}\|u\|_{L^{p_2}(\mathbb{R}^n)}^{\frac{p-\alpha}{p}}\right)^q}{\|u\|_{L^q(\mathbb{R}^n)}^q}\right) \\ &\le \frac{q\alpha}{p-q}\ln\left(\frac{B\|(-\triangle)^{s/2}u\|_{L^{p_1}(\mathbb{R}^n)}}{\|u\|_{L^q(\mathbb{R}^n)}}\right).\end{aligned}$$

Note that

$$\alpha\left(\frac{1}{p_1} - \frac{s}{n}\right) + (p-\alpha)\frac{1}{q} = 1 \quad \text{implies} \quad \frac{q\alpha}{p-q} = \frac{1}{\frac{1}{q} + \frac{s}{n} - \frac{1}{p_1}}.$$

Thus, (2.6) holds. □

When $q = p_1$, the previous logarithmic Gagliardo–Nirenberg inequalities imply the following logarithmic Sobolev inequalities for fractional Laplacian.

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Corollary 2.6. *For any $0 < s < n$, $1 < p < n/s$, $u \in \dot{W}^{s,p}(\mathbb{R}^n)$ such that $\|u\|_{L^p(\mathbb{R}^n)} = 1$, we have*

$$\exp\left(\frac{s}{n}\int_{\mathbb{R}^n} |u(x)|^p \ln |u(x)|^p dx\right) \le B\|(-\triangle)^{s/2}u\|_{L^p(\mathbb{R}^n)} \tag{2.7}$$

with B given by (2.3).

Remark 2.7. (i) If the best constant for (2.7) is defined as

$$D = \sup\left\{\frac{\exp\left(\frac{s}{n}\int_{\mathbb{R}^n} |u(x)|^p \ln |u(x)|^p dx\right)}{\|(-\triangle)^{s/2}u\|_{L^p(\mathbb{R}^n)}} : u \in \dot{W}^{s,p}(\mathbb{R}^n),\ \|u\|_{L^p(\mathbb{R}^n)} = 1\right\},$$

then D satisfies $D \le B$. The exact form of D is unknown. For $p = 2$, an upper bound estimate for D was given by Cotsiolis and Tavoularis in [15].

(ii) When $s = 1$, it is well known that the sharp constant for the logarithmic Sobolev inequality for $p = 2$ is $(\frac{2}{ne\pi})^{1/2}$ which is the asymptotic of the sharp constant for the corresponding Sobolev inequality. So, it is reasonable to guess that the best constant D for (2.7) is the asymptotic of the best constant B for fractional Sobolev inequalities (2.2). Recalling the Stirling formula

$$\Gamma(s+1) \sim \sqrt{2\pi}s^{s+\frac{1}{2}}, \quad \text{as } s \longrightarrow \infty,$$

we get

$$\left(\frac{\Gamma(n)}{\Gamma(n/2)}\right)^{s/n} \sim \left(\frac{2n}{e}\right)^{s/2} \quad \text{and} \quad \frac{\Gamma((n-s)/2)}{\Gamma((n+s)/2)} \sim \left(\frac{2}{n}\right)^{s}$$

as $n \longrightarrow \infty$. Then, $B \sim (\frac{2}{ne\pi})^{s/2}$ as $n \longrightarrow \infty$. So, we guess $D = (\frac{2}{ne\pi})^{s/2}$.

2.3. *Logarithmic Sobolev trace inequalities*

When $p = 2$, $\dot{W}^{s,2}(\mathbb{R}^n) = \dot{H}^s(\mathbb{R}^n)$ is endowed the norm

$$\|u\|_{\dot{H}^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |2\pi\xi|^{2s}|\widehat{u}(\xi)|^2 d\xi\right)^{1/2},$$

which is equivalent to $(\int_{\mathbb{R}^{n+1}_+} |\nabla_{x,t}h(x,t)|^2 t^{1-2s} dxdt)^{1/2}$ up to a constant when $0 < s < 1$ (see e.g. [23,24]). Here $h(x,t)$ is the extension of $u(x)$ defined through the equation

$$\begin{cases} h(x,0) = u(x), \\ \triangle_x h + \dfrac{1-2s}{t}h_t + h_{tt} = 0. \end{cases} \tag{2.8}$$

So, when $0 < s < 1$, if we use $(\int_{\mathbb{R}^{n+1}_+} |\nabla_{x,t}h(x,t)|^2 t^{1-2s} dxdt)^{1/2}$ to replace the right hand side of (2.7) up to a constant, then we can think (2.7) as a logarithmic Sobolev trace inequality. Note that $(\int_{\mathbb{R}^{n+1}_+} |\nabla_{x,t}h(x,t)|^2 t^{1-2s} dxdt)^{1/2}$ is not the L^2-norm of the fractional Laplacian of a function on $\mathbb{R}^{n+1}$. We will deduce a logarithmic Sobolev trace inequality for fractional Laplacian in terms of the restriction $\tau_k u$ of $u \in \dot{H}^s(\mathbb{R}^n)$ to the $n-k$ dimensional hyperplane given by

$$(\tau_k u)(x_1, x_2, \ldots, x_{n-k}) = u(x_1, x_2, \ldots, x_{n-k}, 0, 0, \ldots, 0)$$

from the following Sobolev trace inequality.

Theorem 2.8 (*Einav and Loss in [37]*). *Let $0 \le k < n$ and $\frac{k}{2} < s < \frac{n}{2}$. For any $u \in \dot{H}^s(\mathbb{R}^n)$, we have*

$$\|\tau_k u\|^2_{L^{\frac{2(n-k)}{n-2s}}(\mathbb{R}^{n-k})} \le E\|u\|_{\dot{H}^s(\mathbb{R}^n)} \tag{2.9}$$

with the best constant

$$E = 2^{-2s}\pi^{-s}\frac{\Gamma(n/2-s)\Gamma(s-k/2)}{\Gamma(s)\Gamma(n/2+s-k)}\left(\frac{\Gamma(n-k)}{\Gamma((n-k)/2)}\right)^{\frac{2s-k}{n-k}}.$$

In the following, we use this Sobolev trace inequality to obtain a logarithmic Sobolev trace inequality.

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Theorem 2.9. *Let k, n be integers with $n \geq 2$, $0 \leq k < n$ and $\frac{k}{2} < s < \frac{n}{2}$. If $u \in \dot{H}^s(\mathbb{R}^n)$ and $\|\tau_k u\|_{L^2(\mathbb{R}^{n-k})} = 1$, then*

$$\exp\left(\int_{\mathbb{R}^{n-k}} |(\tau_k u)(x)|^2 \ln |\tau_k u(x)| dx\right) \leq F \|u\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{n-k}{4s-2k}} \tag{2.10}$$

with

$$F = \left(2^{-2s}\pi^{-s}\frac{\Gamma(n/2-s)\Gamma(s-k/2)}{\Gamma(s)\Gamma(n/2+s-k)}\right)^{\frac{n-k}{4s-2k}} \left(\frac{\Gamma(n-k)}{\Gamma((n-k)/2)}\right)^{\frac{1}{2}}.$$

Proof. Let $p = \frac{(n-k)(q-2)}{2s-k}$, $2 < q < \frac{2(n-k)}{n-2s}$. Applying the Hölder inequality, we have

$$\begin{aligned}\|\tau_k u\|^q_{L^q(\mathbb{R}^{n-k})} &= \int_{\mathbb{R}^{n-k}} |\tau_k u(x)|^p |\tau_k u(x)|^{q-p} dx \\ &\leq \|\tau_k u\|^p_{L^{\frac{2(n-k)}{n-2s}}(\mathbb{R}^{n-k})} \left(\int_{\mathbb{R}^{n-k}} |(\tau_k u)(x)|^2\right)^{1-\frac{p(n-2s)}{2(n-k)}}.\end{aligned}$$

Using the fact $\|\tau_k u\|_{L^2(\mathbb{R}^{n-k})} = 1$ and (2.9), we get

$$\begin{aligned}\left(\int_{\mathbb{R}^{n-k}} |\tau_k u(x)|^{q-2}|\tau_k u(x)|^2 dx\right)^{\frac{1}{q-2}} &= \left(\int_{\mathbb{R}^{n-k}} |\tau_k u(x)|^q dx\right)^{\frac{1}{q-2}} \\ &\leq (E\|u\|_{\dot{H}^s(\mathbb{R}^n)})^{\frac{n-k}{4s-2k}}.\end{aligned} \tag{2.11}$$

Since $|\tau_k u|^2 dx$ can be treated as a probability measure on $\mathbb{R}^{n-k}$, Jensen's inequality implies

$$\begin{aligned}\exp\left(\int_{\mathbb{R}^{n-k}} |\tau_k u(x)|^2 \ln(|\tau_k u(x)|) dx\right) &= \left(\exp\left(\int_{\mathbb{R}^{n-k}} |\tau_k u(x)|^2 \ln(|\tau_k u(x)|^{q-2}) dx\right)\right)^{\frac{1}{q-2}} \\ &\leq \left(\int_{\mathbb{R}^{n-k}} |\tau_k u(x)|^{q-2}|\tau_k u(x)|^2 dx\right)^{\frac{1}{q-2}}.\end{aligned} \tag{2.12}$$

So, (2.11) and (2.12) give us (2.10). □

Remark 2.10. Xiao in [24] proved that, for $u \in \dot{H}^s(\mathbb{R}^{n-1})$ with $\|u\|_{L^2(\mathbb{R}^{n-1})} = 1$,

$$\exp\left(\int_{\mathbb{R}^{n-1}} |u(x)|^2 \ln |u(x)| dx\right) \leq G\left(\int_{\mathbb{H}^n} |\nabla(e^{-(-\triangle)^{1/2}t}u)(x,t)|^2 t^{1-2s} dx dt\right)^{\frac{n-1}{4s}}$$

with

$$G = \left(\frac{2^{1-4s}}{\pi^s\Gamma(2-2s)}\frac{\Gamma((n-1)/2-s)}{\Gamma((n-1)/2+s)}\right)^{\frac{n-1}{4s}} \left(\frac{\Gamma(n-1)}{\Gamma((n-1)/2)}\right)^{\frac{1}{2}}.$$

Note that for the case $k = 1$, our constant differs from Xiao's by a factor $1/2$. This is due to the fact that $(e^{-(-\triangle)^{1/2}t}u)(x,t)$ is only defined on $\mathbb{H}^n = \{(x,t) : x \in \mathbb{R}^{n-1}, t > 0\}$.

3. Fractional Hardy inequalities

In this section, we prove the Hardy inequalities for fractional Laplacian in Lorentz spaces and an upper bound for the constant.

Theorem 3.1. *Let $1 < p < \infty$, $0 < s < n/q$, and $1 \leq q \leq \infty$. There holds*

$$\left\|\frac{u(x)}{|x|^s}\right\|_{L^{p,q}(\mathbb{R}^n)} \leq H\|(-\triangle)^{s/2}u(x)\|_{L^{p,q}(\mathbb{R}^n)}, \tag{3.1}$$

with

$$H \leq \frac{3p^2\omega_{n-1}}{(n-ps)(p-1)}$$

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if $1 \le q < \infty$, *and*

$$H \le e^{1/e} \left(\frac{\omega_{n-1} p^2}{s(p-1)^2 + \frac{np(p-1)}{n-ps}} \right) K_{s,n}$$

if $q = \infty$.

Proof. Recall a generalized version of Hölder's inequality: if $1 \le p_1, p_2, q_1, q_2 \le \infty$, then

$$\|fg\|_{L^{p,q}(\mathbb{R}^n)} \le \frac{p}{p-1} \|f\|_{L^{p_1,q_1}(\mathbb{R}^n)} \|g\|_{L^{p_2,p_2}(\mathbb{R}^n)}$$

for any $f \in L^{p_1,q_1}(\mathbb{R}^n)$ and $g \in L^{p_2,p_2}(\mathbb{R}^n)$ with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

The proof of this inequality can be found in [34].

We take $g(x) = |x|^{-s}$ and it is easy to see that $g(x) \in L^{n/s,\infty}(\mathbb{R}^n)$ with $\|g\|_{L^{n/s,\infty}(\mathbb{R}^n)} = (\frac{\omega_{n-1}}{n})^{s/n}$. Here ω_{n-1} is the surface area of the unit ball in $\mathbb{R}^n$. So, we get

$$\left\| \frac{u(x)}{|x|^s} \right\|_{L^{p,q}(\mathbb{R}^n)} \le \frac{p}{p-1} \left(\frac{\omega_{n-1}}{n} \right)^{s/n} \|u(x)\|_{L^{\frac{np}{n-ps},q}(\mathbb{R}^n)}$$

with $1 \le q \le \infty$. It follows from [26, Theorem 2.1, Remark 2.4] that

$$\|u(x)\|_{L^{\frac{np}{n-ps},q}(\mathbb{R}^n)} \le c\|(-\triangle)^{s/2} u(x)\|_{L^{p,q}(\mathbb{R}^n)}$$

for any $u \in L^{p,q}(\mathbb{R}^n)$ with $1 \le q \le \infty$, $1 < p < \infty$, and $0 < s < n/p$. Here the sharp constant c satisfies

$$c \le \frac{3np}{n-ps} \left(\frac{\omega_{n-1}}{n} \right)^{\frac{n-s}{n}}$$

if $1 \le q < \infty$, and

$$c \le e^{1/e} \left(\frac{np}{s(p-1) + \frac{np}{n-ps}} \right) K_{s,n} \left(\frac{\omega_{n-1}}{n} \right)^{\frac{n-s}{n}}$$

if $q = \infty$, where $K_{s,n} = \frac{\Gamma(\frac{n-s}{2})}{\pi^{n/2} 2^s \Gamma(s/2)}$. Thus, we have (3.1). □

Remark 3.2. When $p = q$ and $0 < s < 1$, (3.1) becomes

$$\left(\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx \right)^{1/p} \le G\|(-\triangle)^{s/2} u(x)\|_{L^p(\mathbb{R}^n)}$$

which has been studied in [27].

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References

[1] H. Hajaiej, L. Molinet, T. Ozawa, B. Wang, Sufficient and necessary conditions for the fractional Gagliardo–Nirenberg inequalities and applications to Navier–Stokes and generalized boson equations, in: T. Ozawa, M. Sugimoto (Eds.), RIMS Kkyroku Bessatsu B26: Harmonic Analysis and Nonlinear Partial Differential Equations, 2011–05, pp. 159–175.
[2] P. Li, Z. Zhai, Well-posedness and regularity of generalized Navier–Stokes equations in some critical Q-spaces, J. Funct. Anal. 259 (2010) 2457–2519.
[3] P. Li, Z. Zhai, Generalized Naiver–Stokes equations with initial data in local Q-type spaces, J. Math. Anal. Appl. 369 (2010) 595–609.
[4] P. Li, Z. Zhai, Riesz transforms on Q-type spaces with application to quasi-geostrophic equation, Taiwanese J. Math. (in press).
[5] C. Miao, B. Yuan, B. Zhang, Well-posedness of the Cauchy problem for the fractional power dissipative equations, Nonlinear Anal. TMA 68 (2008) 461–484.
[6] J. Wu, The generalized incompressible Navier–Stokes equations in Besov spaces, Dynamics. PDE 1 (2004) 381–400.
[7] J. Wu, Lower bounds for an integral involving fractional Laplacians and the generalized Navier–Stokes equations in Besov spaces, Commun. Math. Phys. 263 (2005) 803–831.
[8] X. Yu, Z. Zhai, Well-posedness for fractional Navier–Stokes equations in the largest critical spaces $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$, Math. Methods Appl. Sci. 35 (2012) 676–683.
[9] Z. Zhai, Strichartz type estimates for fractional heat equations, J. Math. Anal. Appl. 356 (2009) 642–658.

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[10] Z. Zhai, Well-posedness for fractional Navier–Stokes equations in critical spaces close to $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$, Dynamics. PDE 7 (2010) 25–44.
[11] Z. Zhai, Global well-posedness for nonlocal fractional Keller–Segel systems in critical Besov spaces, Nonlinear Analysis: TMA 72 (2010) 3173–3189.
[12] L. Caffarelli, A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Ann. of Math. 171 (2010) 1903–1930.
[13] R.L. Frank, E. Lenzmann, Uniqueness and nondegeneracy of ground states for $(-\triangle)^s Q + Q - Q^{\alpha+1} = 0$ in $\mathbb{R}$, Acta Math. (in press).
[14] A. Cotsiolis, N.K. Tavoularis, Sharp Sobolev type inequalities for higher fractional derivatives, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 801–804.
[15] A. Cotsiolis, N.K. Tavoularis, Best constants for Sobolev inequalities for higher order fractional derivatives, J. Math. Anal. Appl. 295 (2004) 225–236.
[16] J. Merker, Generalizations of logarithmic Sobolev inequalities, Discrete Contin. Dyn. Syst. Ser. S 1 (2) (2008) 329–338.
[17] J. Merker, Regularity of solutions to doubly nonlinear diffusion equations, Electron. J. Differ. Equ. Conf. 17 (2009) 185–195.
[18] A. Cotsiolis, N.K. Tavoularis, On logarithmic Sobolev inequalities for higher order fractional derivatives, C. R. Acad. Sci. Paris, Ser. I 340 (2005) 205–208.
[19] L. Gross, Logarithmic Sobolev inequalities, Amer. J. Math. 97 (1975) 1061–1083.
[20] M. Del Pino, J. Dolbeault, The optimal Euclidian L_p-Sobolev logarithmic inequality, J. Funct. Anal. 197 (2003) 151–161.
[21] E.A. Carlen, M. Loss, Logarithmic Sobolev inequalities and spectral gaps, in: Recent Advances in the Theory and Applications of Mass Transport, in: Contemp. Math., vol. 353, Amer. Math. Soc., Providence, RI, 2004, pp. 53–60.
[22] W. Beckner, M. Pearson, On sharp Sobolev embedding and the logarithmic Sobolev inequalities, Bull. London Math. Soc. 30 (1998) 80–84.
[23] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Commun. Partial Differ. Equations 32 (8) (2007) 1245–1260.
[24] J. Xiao, A sharp Sobolev trace inequality for the fractional-order derivatives, Bull. Sci. Math. 130 (2006) 87–96.
[25] Y.J. Park, Logarithmic Sobolev trace inequality, Proc. Am. Math. Soc. 132 (2004) 2075–2083.
[26] J. Xiao, Z. Zhai, Fractional Sobolev, Moser–Trudinger, Morrey–Sobolev inequalities under Lorentz norms, J. Math. Sci. 166 (2010) 357–376.
[27] R. Frank, R. Seringer, Nonlinear ground state representation and sharp Hardy inequalities, J. Funct. Anal. 255 (2008) 3407–3430.
[28] K. Bogdan, B. Dyda, The best constant in a fractional Hardy inequality, Math. Nachr. 284 (2011) 629–638.
[29] B. Dyda, A fractional order Hardy inequalities, Ill. J. Math. 48 (2004) 575–588.
[30] R. Frank, R. Seiringer, Sharp Fractional Hardy Inequalities in Half-Space, in: Around Sobolev Spaces, Springer, New York, 2010, pp. 161–167.
[31] M. Loss, C. Sloane, Hardy inequalities for fractional integrals on general domains, J. Func. Anal. 259 (2010) 1369–1379.
[32] E. Ostrovsky, L. Sirota, Necessary conditions for fractional Hardy–Sobolev's inequalities, arXiv:1108.1387v1.
[33] R.A. Hunt, On $L(p,q)$ spaces, Enseign. Math. 12 (1966) 249–276.
[34] R. O'Neil, Convolution operators on $L(p,q)$ spaces, Duke Math. J. 30 (1963) 129–142.
[35] E.M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Univ. Press, 1971.
[36] E.H. Lieb, Sharp constants in the Hardy–Littlewood–Sobolev inequalities and related inequalities, Ann. of Math. 118 (1983) 349–374.
[37] A. Einav, M. Loss, Sharp trace inequalities for fractional Laplacians, arXiv:1105.4079v1.

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