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A note on the energy conservation of the ideal MHD equations

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Abstract

By exploring the special structure of the nonlinear terms in the ideal MHD equations, we obtain new sufficient conditions for the conservation of energy and cross-helicity. These conditions improve those given in Caflisch *et al* (1997 *Commun. Math. Phys.* **184** 443–55) and Kang-Lee (2007 *Nonlinearity* **20** 2681–9).

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1. Introduction

Turbulence is a ubiquitous fluid mechanical phenomenon bearing great scientific and engineering importance. One major character of turbulent fluids is that the dissipation mechanism is significantly enhanced. For example, in hydrodynamics, the energy dissipation rate at high Reynolds numbers is observed to be approximately independent of the coefficient of viscosity. In magneto-hydrodynamics, a similar observation has also been made for the dissipation of energy at high Reynolds and magnetic Reynolds numbers [Bis03].

On the other hand, motivated by the fact that turbulent flows are highly irregular, it has been conjectured that weak solutions to the equations of the corresponding ideal fluids (for example the incompressible Euler equations and the ideal MHD equations) are promising candidates for the rigorous mathematical description of general turbulence. Taking into account the aforementioned observation on energy dissipation, a natural first step in the investigation of this conjecture is to study necessary and sufficient conditions involving the regularity of the solutions for the conservations of various physical quantities. In the context of hydrodynamics, this leads to the famous Onsager's conjecture [Ons49] which has been of great interest in the past 20 years [CCFS08, DLSJ07a, DLSJ07b, DR00, CET94, Eyi94, Eyi08, ES06, Sch93, Shn97]. In the context of magneto-hydrodynamics, sufficient conditions have been obtained in [CKS97] and [KL07] by directly applying the methods developed in the study of Onsager's conjecture to the ideal MHD equations.

In this paper we prove new sufficient conditions for the conservation of energy and cross-helicity of the ideal MHD equations. Instead of directly applying the methods developed in the study of Onsager's conjecture, we first explore the special structures of the nonlinear terms in the equations. As a consequence, our new conditions are weaker than those in [CKS97, KL07].

The ideal MHD equations in $\mathbb{R}^3$ read as follows:

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p - \tfrac{1}{2}\nabla|\boldsymbol{b}|^2 + \boldsymbol{b}\cdot\nabla\boldsymbol{b}, \tag{1}$$

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{b} = \boldsymbol{b}\cdot\nabla\boldsymbol{u}, \tag{2}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{b} = 0, \tag{3}$$

where $\boldsymbol{u}$ is the velocity, $\boldsymbol{b}$ the magnetic field and p the pressure.

For solutions of (1)–(3) with enough regularity to justify integration by parts, it is clear that the following quantities are conserved:

– energy:

$$\int_{\mathbb{R}^3} |\boldsymbol{u}(\boldsymbol{x},t)|^2 + |\boldsymbol{b}(\boldsymbol{x},t)|^2 \,\mathrm{d}\boldsymbol{x}; \tag{4}$$

– magnetic helicity:

$$\int_{\mathbb{R}^3} \boldsymbol{a}(\boldsymbol{x},t)\cdot\boldsymbol{b}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}, \tag{5}$$

where $\boldsymbol{a}$ is the vector potential of $\boldsymbol{b}$, that is $\boldsymbol{b} = \nabla\times\boldsymbol{a}$;

– cross-helicity:

$$\int_{\mathbb{R}^3} \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{b}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}. \tag{6}$$

The conservation of these quantities has significant physical meanings and reflects geometric and topological properties of the flow. In the following we will study whether these conservations still hold for less regular solutions. To do this, we need the notion of weak solutions, which is as follows.

Definition 1 (Weak solutions). *Let* $\boldsymbol{u},\boldsymbol{b}\in C_w([0,T];L^2(\mathbb{R}^3))$*, that is*

$$\int_{\mathbb{R}^3}\boldsymbol{u}\cdot\phi\,\mathrm{d}\boldsymbol{x},\qquad \int_{\mathbb{R}^3}\boldsymbol{b}\cdot\phi\,\mathrm{d}\boldsymbol{x}\in C([0,T]) \tag{7}$$

for any test function $\phi = (\phi_1,\phi_2,\phi_3)$ *with* $\phi_i\in C_0^\infty(\mathbb{R}^3)$, $i=1,2,3$. *The pair* $(\boldsymbol{u},\boldsymbol{b})$ *is called a weak solution of the ideal MHD equations (1)–(3) if*

(i) both $\boldsymbol{u},\boldsymbol{b}$ *are divergence free in the sense of distributions and*

(ii) for every $\psi,\boldsymbol{\eta}\in C^1([0,T];\mathcal{S}(\mathbb{R}^n))$*, where* $\mathcal{S}(\mathbb{R}^n)$ *is the space of rapidly decreasing functions (also called the Schwartz space), with* $\nabla\cdot\psi = \nabla\cdot\boldsymbol{\eta} = 0$ *and every* $t\in[0,T]$ *we have*

$$(\boldsymbol{u}(t),\psi(t)) - (\boldsymbol{u}(0),\psi(0)) - \int_0^t (\boldsymbol{u}(s),\partial_s\psi(s))\,\mathrm{d}s = \int_0^t b(\boldsymbol{u},\psi,\boldsymbol{u})(s) - b(\boldsymbol{b},\psi,\boldsymbol{b})\,\mathrm{d}s, \tag{8}$$

$$(\boldsymbol{b}(t),\boldsymbol{\eta}(t)) - (\boldsymbol{b}(0),\boldsymbol{\eta}(0)) - \int_0^t (\boldsymbol{b}(s),\partial_s\boldsymbol{\eta}(s))\,\mathrm{d}s = \int_0^t c(\boldsymbol{u},\boldsymbol{b},\boldsymbol{\eta})\,\mathrm{d}s, \tag{9}$$

where

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_{\mathbb{R}^3} \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}, \tag{10}$$

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_{\mathbb{R}^3} \boldsymbol{u} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x}, \tag{11}$$

$$c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_{\mathbb{R}^3} (\boldsymbol{u} \times \boldsymbol{v}) \cdot (\nabla \times \boldsymbol{w}) \, \mathrm{d}\boldsymbol{x}. \tag{12}$$

For a weak solution $(\boldsymbol{u}, \boldsymbol{b})$, it is shown in [CKS97] that

– the energy (4) is conserved if $\boldsymbol{u} \in C([0, T], B^{\alpha_1}_{3,\infty})$, $\boldsymbol{b} \in C([0, T], B^{\alpha_2}_{3,\infty})$, $\alpha_1 > 1/3$ and $\alpha_1 + 2\alpha_2 > 1$;
– the magnetic helicity (5) is conserved if $\boldsymbol{u} \in C([0, T], B^{\alpha_1}_{3,\infty})$, $\boldsymbol{b} \in C([0, T], B^{\alpha_2}_{3,\infty})$ and $\alpha_1 + 2\alpha_2 > 0$.

Here $B^{\alpha}_{3,\infty}$ denotes a class of Besov spaces which refines the Hölder-α regularity (see appendix for definitions).

More recently, sharper sufficient conditions have been obtained in [KL07]:

– the energy (4) is conserved if $\boldsymbol{u} \in L^3([0, T], B^{\alpha_1}_{3,c(\mathbb{N})})$, $\boldsymbol{b} \in L^3([0, T], B^{\alpha_2}_{3,c(\mathbb{N})})$, $\alpha_1 \geqslant 1/3$ and $\alpha_1 + 2\alpha_2 \geqslant 1$;
– the magnetic helicity (5) is conserved if $\boldsymbol{u}, \boldsymbol{b} \in L^3([0, T], L^3)$;
– the cross-helicity (6) is conserved if $\boldsymbol{u} \in L^3([0, T], B^{\alpha_1}_{3,c(\mathbb{N})})$, $\boldsymbol{b} \in L^3([0, T], B^{\alpha_2}_{3,c(\mathbb{N})})$, $\alpha_2 \geqslant 1/3$ and $\alpha_1 + 2\alpha_2 \geqslant 1$

The spaces $B^{\alpha}_{3,c(\mathbb{N})}$ (see the appendix for definition) satisfy $B^{\alpha+\varepsilon}_{3,\infty} \subset B^{\alpha}_{3,c(\mathbb{N})} \subset B^{\alpha}_{3,\infty}$ for any $\varepsilon > 0$.

The above results are obtained by direct applications of the methods developed in the study of Onsager's conjecture for the 3D incompressible Euler equations. In this short note, we obtain sharper sufficient conditions by exploring the special properties of the MHD nonlinearity. In short, inspection of (1)–(3) reveals that the magnetic field vector $\boldsymbol{b}$ appears linearly in its equation (2). This observation is the key to the relaxation of conditions on $\boldsymbol{b}$.

2. Summary of main results

To better present our main results, we introduce the following regularity conditions which are in the same spirit as those introduced in [Shv08] and [DR00].

(C1) $\boldsymbol{v}$ is said to satisfy condition (C1) with regularity α if

$$\lim_{\boldsymbol{y}\to\boldsymbol{0}} \frac{\left(\int_0^T \int_{\mathbb{R}^3} |\boldsymbol{v}(\boldsymbol{x}-\boldsymbol{y}, t) - \boldsymbol{v}(\boldsymbol{x}, t)|^3 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t\right)^{1/3}}{|\boldsymbol{y}|^{3\alpha}} \equiv \lim_{\boldsymbol{y}\to 0} \frac{\|\delta_{\boldsymbol{y}} \boldsymbol{v}\|_{L^3([0,T],L^3)}}{|\boldsymbol{y}|^{\alpha}} = 0, \tag{13}$$

where $\delta_{\boldsymbol{y}} \boldsymbol{v}(\boldsymbol{x}, t) \equiv \boldsymbol{v}(\boldsymbol{x}-\boldsymbol{y}, t) - \boldsymbol{v}(\boldsymbol{x}, t)$.

(C2) $\boldsymbol{v}$ is said to satisfy condition (C2) with regularity α if

$$\lim_{q\nearrow\infty} \int_0^T 2^{3\alpha q} \|\Delta_q \boldsymbol{v}\|^3_{L^3} \to 0. \tag{14}$$

Similarly, one can introduce the following.

(B1) $\boldsymbol{v}$ is said to satisfy condition (B1) with regularity α if

$$\frac{\left(\int_0^T \int_{\mathbb{R}^3} |\boldsymbol{v}(\boldsymbol{x}-\boldsymbol{y}, t) - \boldsymbol{v}(\boldsymbol{x}, t)|^3 \mathrm{d}\boldsymbol{x} \, \mathrm{d}t\right)^{1/3}}{|\boldsymbol{y}|^{\alpha}} \equiv \frac{\|\delta_{\boldsymbol{y}} \boldsymbol{v}\|_{L^3([0,T],L^3)}}{|\boldsymbol{y}|^{\alpha}} \tag{15}$$

is uniformly bounded in $\boldsymbol{y}$.

(B2) $\boldsymbol{v}$ is said to satisfy condition (B2) with regularity α if

$$\int_0^T 2^{3\alpha q} \|\Delta_q \boldsymbol{v}\|_{L^3}^3 \tag{16}$$

is uniformly bounded in q.

We will prove the following lemma in the appendix.

Lemma 1. *Let $v \in L^3([0, T], L^3)$. Then*

(a) v satisfies (B2) with regularity $\alpha \Longleftrightarrow v$ satisfies (B1) with regularity α;
(b) v satisfies (C2) with regularity $\alpha \Longleftrightarrow v$ satisfies (C1) with regularity α;
(c) $v \in L^3([0, T], B^{\alpha}_{3,\infty}) \Longrightarrow v$ satisfies (B2) with regularity α;
(d) $v \in L^3([0, T], B^{\alpha}_{3,c(\mathbb{N})}) \Longrightarrow v$ satisfies (C2) with regularity α.

Now we are ready to state the main results.

Theorem 1 (Conservation of energy). *Let $(\boldsymbol{u}, \boldsymbol{b})$ be a weak solution of the ideal MHD equations in $\mathbb{R}^3$. The energy*

$$\int_{\mathbb{R}^3} |\boldsymbol{u}(\boldsymbol{x}, t)|^2 + |\boldsymbol{b}(\boldsymbol{x}, t)|^2 \, \mathrm{d}\boldsymbol{x} \tag{17}$$

is conserved if $\boldsymbol{u}$ satisfies (C1), or equivalently (C2), with regularity α_1, $\boldsymbol{b}$ satisfies (B1), or equivalently (B2), with regularity α_2 and furthermore $\alpha_1 \geqslant 1/3$, $\alpha_1 + 2\alpha_2 \geqslant 1$.

Theorem 2 (Conservation of cross-helicity). *Let $(\boldsymbol{u}, \boldsymbol{b})$ be a weak solution of the ideal MHD equations in $\mathbb{R}^3$. The cross-helicity*

$$\int_{\mathbb{R}^3} \boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{b}(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x} \tag{18}$$

is conserved if $\boldsymbol{u}$ satisfies (B1), or equivalently (B2), with regularity α_1, $\boldsymbol{b}$ satisfies (C1), or equivalently (C2), with regularity α_2 and furthermore $\alpha_2 \geqslant 1/3$, $\alpha_1 + 2\alpha_2 \geqslant 1$.

Remark 1. In light of lemma 1(c) and the fact that $B^{\alpha}_{3,c(\mathbb{N})} \subset B^{\alpha}_{3,\infty}$ (see the appendix), we see that the conditions in theorems 1 and 2 are indeed weaker than the corresponding conditions in [KL07], where both $\boldsymbol{u}$ and $\boldsymbol{b}$ are required to be in spaces of the $B^{\alpha}_{3,c(\mathbb{N})}$-type.

Remark 2. In light of theorems 1 and 2, the theory of weak solutions with critical regularity, that is both $\boldsymbol{u}, \boldsymbol{b}$ satisfying only (B1)/(B2) but not (C1)/(C2), would be most relevant to magneto-hydrodynamical turbulence, as these are the solutions that may 'just fail to conserve energy'. Furthermore, the study of how the regularity of classical solutions deteriorates and finally ceases to satisfy the conditions in theorems 1 and 2 may shed light on the important problem of understanding the transition to turbulence. Unfortunately, such theories are still missing. To the best of the author's knowledge, the least regular function space for $(\boldsymbol{u}, \boldsymbol{b})$ in which the local existence and uniqueness are guaranteed is $B^1_{\infty,1}$ (see the appendix for definition) obtained in [MY06]. It can be easily shown that any pair $(\boldsymbol{u}, \boldsymbol{b}) \in B^1_{\infty,1}$ satisfies (C1), and therefore such solutions conserve both energy and cross-helicity.

3. Proof of theorem 1

Let $\varphi \in \mathcal{S}$ be radially symmetric with $\int_{\mathbb{R}^3} \varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1$. We can define the mollifiers in $\mathbb{R}^3$:

$$\varphi^\varepsilon(\boldsymbol{x}) \equiv \frac{1}{\varepsilon^3}\varphi\left(\frac{\boldsymbol{x}}{\varepsilon}\right). \tag{19}$$

For any (vector-valued) function $\boldsymbol{v}$, we define its mollification

$$\boldsymbol{v}^\varepsilon(\boldsymbol{x}) \equiv (\varphi^\varepsilon * \boldsymbol{v})(\boldsymbol{x}). \tag{20}$$

Now taking $\boldsymbol{\psi} = \int_{\mathbb{R}^3} \varphi^\varepsilon(\boldsymbol{y}-\boldsymbol{x})\boldsymbol{u}^\varepsilon(\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y}$ and $\boldsymbol{\eta} = \int_{\mathbb{R}^3} \varphi^\varepsilon(\boldsymbol{y}-\boldsymbol{x})\boldsymbol{b}^\varepsilon(\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y}$ in (8) and (9), and integrating over $\mathbb{R}^3$ and then from 0 to t, we obtain

$$E^\varepsilon(t) - E^\varepsilon(0) = \int_0^t \int_{\mathbb{R}^3} (\boldsymbol{u}\otimes\boldsymbol{u})^\varepsilon : \nabla\boldsymbol{u}^\varepsilon - (\boldsymbol{b}\otimes\boldsymbol{b})^\varepsilon : \nabla\boldsymbol{u}^\varepsilon + (\boldsymbol{u}\times\boldsymbol{b})^\varepsilon \cdot (\nabla\times\boldsymbol{b}^\varepsilon)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \tag{21}$$

where $M : N \equiv \mathrm{Tr}(MN)$ for two matrices M, N and

$$E^\varepsilon(s) \equiv \int_{\mathbb{R}^3} (|\boldsymbol{u}^\varepsilon(\boldsymbol{x},s)|^2 + |\boldsymbol{b}^\varepsilon(\boldsymbol{x},s)|^2)\,\mathrm{d}\boldsymbol{x}. \tag{22}$$

Since $E^\varepsilon(s) \to E(s)$ as $\varepsilon \searrow 0$, energy conservation is guaranteed as long as the right-hand side of (21) vanishes as $\varepsilon \searrow 0$.

Before presenting the details of the proof, we would like to emphasize the key observation that at least one $\boldsymbol{u}$-term is involved in each of the three right-hand side terms of (21).

To simplify the presentation, we consider the following general trilinear form:

$$I^\varepsilon(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \equiv \int_0^t \int_{\mathbb{R}^3} (\boldsymbol{u}\circ\boldsymbol{v})^\varepsilon \circ (D\boldsymbol{w})^\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \tag{23}$$

Here $\circ$ denotes a generic multiplication and D denotes a generic differentiation. For example, the last term on the RHS of (21) can be written as $I^\varepsilon(\boldsymbol{u},\boldsymbol{b},\boldsymbol{b})$ with the first $\circ$ denoting the cross product, the second $\circ$ denoting the dot product and D denoting the curl operator. As we will see soon, only the orders of multiplications and differentiations matter here, the exact details do not.

We now study the conditions on divergence free vector fields $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ which will guarantee $\lim_{\varepsilon\searrow 0} I^\varepsilon(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = 0$. Once this is established, the proof of the theorem becomes straightforward.

We recall a beautiful pointwise identity discovered in [CET94]:

$$(\boldsymbol{u}\circ\boldsymbol{v})^\varepsilon = \boldsymbol{u}^\varepsilon\circ\boldsymbol{v}^\varepsilon + r^\varepsilon(\boldsymbol{u},\boldsymbol{v}) - (\boldsymbol{u}-\boldsymbol{u}^\varepsilon)\circ(\boldsymbol{v}-\boldsymbol{v}^\varepsilon), \tag{24}$$

where

$$r^\varepsilon(\boldsymbol{u},\boldsymbol{v})(\boldsymbol{x}) \equiv \int_{\mathbb{R}^3} \varphi^\varepsilon(\boldsymbol{y})[(\boldsymbol{u}(\boldsymbol{x}-y) - \boldsymbol{u}(\boldsymbol{x}))\circ(\boldsymbol{v}(\boldsymbol{x}-\boldsymbol{y}) - \boldsymbol{v}(\boldsymbol{x}))]\,\mathrm{d}\boldsymbol{y}. \tag{25}$$

Substituting (24) into (23), we obtain

$$\begin{aligned} I^\varepsilon(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) &= \int_0^t \int_{\mathbb{R}^3} (\boldsymbol{u}^\varepsilon\circ\boldsymbol{v}^\varepsilon)\circ(D\boldsymbol{w}^\varepsilon)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^t \int_{\mathbb{R}^3} r^\varepsilon(\boldsymbol{u},\boldsymbol{v})\circ(D\boldsymbol{w}^\varepsilon)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &\quad - \int_0^t \int_{\mathbb{R}^3} (\boldsymbol{u}-\boldsymbol{u}^\varepsilon)\circ(\boldsymbol{v}-\boldsymbol{v}^\varepsilon)\circ(D\boldsymbol{w}^\varepsilon)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &\equiv I_1^\varepsilon(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) + I_2^\varepsilon(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) - I_3^\varepsilon(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}). \end{aligned} \tag{26}$$

We discuss them one by one.

- $I_1^\varepsilon(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$. This term is the only one that depends on the specific structure of the multiplications and derivatives. It does not vanish in general, but one can easily check that for all the three terms on the RHS of (21), $I_1^\varepsilon \equiv 0$ for all ε.
- $I_2^\varepsilon(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$. By Hölder's inequality we obtain

$$|I_2^\varepsilon(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant \int_0^t \|r^\varepsilon(\boldsymbol{u}, \boldsymbol{v})\|_{L^{3/2}} \|D\boldsymbol{w}^\varepsilon\|_{L^3}. \tag{27}$$

Using (25), Minkowski's inequality and Hölder's inequality, we have

$$\|r^\varepsilon(\boldsymbol{u}, \boldsymbol{v})\|_{L^{3/2}} \leqslant \int_{\mathbb{R}^3} |\varphi^\varepsilon(\boldsymbol{y})| \|\delta_{\boldsymbol{y}}\boldsymbol{u}\|_{L^3} \|\delta_{\boldsymbol{y}}\boldsymbol{v}\|_{L^3}\, \mathrm{d}\boldsymbol{y}, \tag{28}$$

where as in the definition of (C1), $\delta_{\boldsymbol{y}}\boldsymbol{v}(\boldsymbol{x}, t) \equiv \boldsymbol{v}(\boldsymbol{x} - \boldsymbol{y}, t) - \boldsymbol{v}(\boldsymbol{x}, t)$.

Now substituting (28) into (27) and using Fubini's theorem, we obtain

$$|I_2^\varepsilon(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant \int_{\mathbb{R}^3} |\varphi^\varepsilon(\boldsymbol{y})| \left[\int_0^t \|\delta_{\boldsymbol{y}}\boldsymbol{u}\|_{L^3} \|\delta_{\boldsymbol{y}}\boldsymbol{v}\|_{L^3} \|D\boldsymbol{w}^\varepsilon\|_{L^3}\, \mathrm{d}t\right] \mathrm{d}\boldsymbol{y}, \tag{29}$$

which in turn gives

$$|I_2^\varepsilon(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant \int_{\mathbb{R}^3} |\varphi^\varepsilon(\boldsymbol{y})| \|\delta_{\boldsymbol{y}}\boldsymbol{u}\|_{L^3([0,T],L^3)} \|\delta_{\boldsymbol{y}}\boldsymbol{v}\|_{L^3([0,T],L^3)} \|D\boldsymbol{w}^\varepsilon\|_{L^3([0,T],L^3)}\, \mathrm{d}\boldsymbol{y}. \tag{30}$$

From this it is clear that

$$\|\delta_{\boldsymbol{y}}\boldsymbol{u}\|_{L^3([0,T],L^3)} \|\delta_{\boldsymbol{y}}\boldsymbol{v}\|_{L^3([0,T],L^3)} \|D\boldsymbol{w}^\varepsilon\|_{L^3([0,T],L^3)} = o(1) \tag{31}$$

as $\boldsymbol{y} \longrightarrow \boldsymbol{0}$ is sufficient for I_2^ε to vanish.
- $I_3^\varepsilon(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$. Direct application of Hölder's inequality gives

$$|I_3^\varepsilon(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant \|\boldsymbol{u}^\varepsilon - \boldsymbol{u}\|_{L^3([0,T],L^3)} \|\boldsymbol{v}^\varepsilon - \boldsymbol{v}\|_{L^3([0,T],L^3)} \|D\boldsymbol{w}^\varepsilon\|_{L^3([0,T],L^3)}. \tag{32}$$

Therefore

$$\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}\|_{L^3([0,T],L^3)} \|\boldsymbol{v}^\varepsilon - \boldsymbol{v}\|_{L^3([0,T],L^3)} \|D\boldsymbol{w}^\varepsilon\|_{L^3([0,T],L^3)} = o(1) \tag{33}$$

as $\varepsilon \searrow 0$ is sufficient for I_3^ε to vanish.

To further simplify the situation, we prove the following lemma, so that it suffices to consider I_2^ε alone.

Lemma 2. *We have*

$$\|\delta_{\boldsymbol{y}}\boldsymbol{v}\|_{L^3([0,T],L^3)} = O(|\boldsymbol{y}|^\alpha) \Longrightarrow \|\boldsymbol{v}^\varepsilon - \boldsymbol{v}\|_{L^3([0,T],L^3)} = O(\varepsilon^\alpha), \tag{34}$$

$$\|\delta_{\boldsymbol{y}}\boldsymbol{v}\|_{L^3([0,T],L^3)} = o(|\boldsymbol{y}|^\alpha) \Longrightarrow \|\boldsymbol{v}^\varepsilon - \boldsymbol{v}\|_{L^3([0,T],L^3)} = o(\varepsilon^\alpha). \tag{35}$$

Proof. Let $M(\boldsymbol{y}) = \|\delta_{\boldsymbol{y}}\boldsymbol{v}\|_{L^3([0,T],L^3)}/|\boldsymbol{y}|^\alpha$. Note that

$$\begin{aligned}
\|\boldsymbol{v}^\varepsilon - \boldsymbol{v}\|_{L^3([0,T],L^3)} &= \left\| \int_{\mathbb{R}^3} \varphi^\varepsilon(\boldsymbol{y})(\boldsymbol{v}(\boldsymbol{x} - \boldsymbol{y}) - \boldsymbol{v}(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{y} \right\|_{L^3([0,T],L^3)} \\
&\leqslant \int_{\mathbb{R}^3} |\varphi^\varepsilon(\boldsymbol{y})| \|\delta_{\boldsymbol{y}}\boldsymbol{v}\|_{L^3([0,T],L^3)}\, \mathrm{d}\boldsymbol{y} \\
&= \int_{\mathbb{R}^3} \frac{1}{\varepsilon^n} \left| \varphi\left(\frac{\boldsymbol{y}}{\varepsilon}\right) \right| M(\boldsymbol{y}) |\boldsymbol{y}|^\alpha\, \mathrm{d}\boldsymbol{y} \\
&= \varepsilon^\alpha \int_{\mathbb{R}^3} |\varphi(z)| |z|^\alpha M(\varepsilon z)\, \mathrm{d}z,
\end{aligned} \tag{36}$$

where we have used Minkowski's inequality and set $\boldsymbol{z} = \boldsymbol{y}/\varepsilon$. The conclusions of the lemma follow. □

In light of lemma 2, $I^\varepsilon(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \longrightarrow 0$ when

$$\|\delta_{\boldsymbol{y}}\boldsymbol{u}\|_{L^3([0,T],L^3)} = O(|\boldsymbol{y}|^{\beta_1}), \quad \|\delta_{\boldsymbol{y}}\boldsymbol{v}\|_{L^3([0,T],L^3)} = O(|\boldsymbol{y}|^{\beta_2}), \quad \|D\boldsymbol{w}^\varepsilon\|_{L^3([0,T],L^3)} = O(|\boldsymbol{y}|^{\beta_3})$$

with $\beta_1 + \beta_2 + \beta_3 \geqslant 0$ and at least one of the Os is o.

Note that the first two conditions are simply conditions (B1) ((C1) when O is replaced by o) with regularity β_1, β_2. The third condition $\|D\boldsymbol{w}^\varepsilon\|_{L^3([0,T],L^3)} = O(|\boldsymbol{y}|^{\beta_3})$ becomes equivalent to (B2) ((C2) when O is replaced by o) when we choose the following particular mollifier φ.

Take a smooth radially symmetric function φ such that its Fourier transform $\mathcal{F}(\varphi) = 0$ for $|\boldsymbol{\xi}| \geqslant 4/3$ and $\mathcal{F}(\varphi) = 1$ for $|\boldsymbol{\xi}| \leqslant 2/3$. For this particular mollifier we have

$$\varphi * \boldsymbol{v} = S_{-1}\boldsymbol{v}, \tag{37}$$

$$\varphi^{\varepsilon_j} * \boldsymbol{v} = S_{j-1}\boldsymbol{v}, \tag{38}$$

$$\varphi^{\varepsilon_{j+1}} * \boldsymbol{v} - \varphi^{\varepsilon_j} * \boldsymbol{v} = \Delta_j\boldsymbol{v}, \tag{39}$$

where $\varepsilon_j = 2^{-j}$ and S_j, Δ_j are Littlewood–Paley decomposition operators (see the appendix).

Standard Littlewood–Paley theory then gives

$$\|D\boldsymbol{w}^{\varepsilon_j}\|_{L^3([0,T],L^3)} = \|D(S_{j-1}\boldsymbol{w})\|_{L^3([0,T],L^3)} = O(\varepsilon_j^{\beta_3}) \tag{40}$$

when $\boldsymbol{w}$ satisfies conditions (B2) ((C2) when O is replaced by o) with regularity $\beta_3 + 1$.

Putting everything together, it is clear that when $\boldsymbol{u}, \boldsymbol{b}$ satisfy the conditions specified in theorem 1, the RHS of (21) scales as

$$o(\varepsilon_j^{3\alpha_1}) + o(\varepsilon_j^{\alpha_1+2\alpha_2}) \longrightarrow 0 \tag{41}$$

as $\varepsilon_j \longrightarrow 0$.

Thus ends the proof of theorem 1.

4. Proof of theorem 2

The proof is almost identical to the proof of theorem 1, as long as we note that conservation holds as long as the following three terms vanish as $\varepsilon \searrow 0$:

$$\int_0^t \int_{\mathbb{R}^3} (\boldsymbol{u} \otimes \boldsymbol{u})^\varepsilon : \nabla \boldsymbol{b}^\varepsilon \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t, \qquad \int_0^t \int_{\mathbb{R}^3} (\boldsymbol{b} \otimes \boldsymbol{b})^\varepsilon : \nabla \boldsymbol{b}^\varepsilon \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t$$

and

$$\int_0^t \int_{\mathbb{R}^3} (\boldsymbol{u} \times \boldsymbol{b})^\varepsilon \cdot (\nabla \times \boldsymbol{u}^\varepsilon)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t.$$

We will not repeat the details here.

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Appendix

The purpose of this appendix is to make this short note as self-contained as possible. We will first review basic facts about Besov spaces and then give a proof to lemma 1.

Littlewood–Paley decomposition and Besov spaces. Let $h \in C_0^\infty(\mathbb{R}^n)$ be defined on the frequency space, radially symmetric and satisfying

$$h(\xi) = \begin{cases} 0 & |\xi| \geqslant 4/3, \\ 1 & |\xi| \leqslant 2/3. \end{cases} \tag{42}$$

Let

$$\chi(\xi) = h\left(\frac{\xi}{2}\right) - h(\xi), \tag{43}$$

so that

$$h(\xi) + \sum_{q=0}^{\infty} \chi(2^{-q}\xi) = 1 \tag{44}$$

for all ξ.

We define the following Fourier multipliers for any distribution f:

$$S_{-1} f \equiv \mathcal{F}^{-1}(h\mathcal{F}(f)), \tag{45}$$

$$\Delta_q f \equiv \mathcal{F}^{-1}(\chi(2^{-q}\xi)\mathcal{F}(f)), \qquad q \in \{0\} \cup \mathbb{N}, \tag{46}$$

$$S_Q f \equiv S_{-1} f + \sum_{0}^{Q} \Delta_q f, \qquad Q \in \{0\} \cup \mathbb{N}. \tag{47}$$

Definition 2 (Besov spaces). *Let $s \in \mathbb{R}$ and $p, r > 1$ be real numbers; the Besov space $B^s_{p,r}$ is determined by the norm*

$$\|f\|_{B^s_{p,r}} \equiv \|S_{-1} f\|_{L^p} + \|(2^{qs}\|\Delta_q f\|_{L^p})_{q\in\mathbb{N}\cup\{0\}}\|_{l^r}. \tag{48}$$

In particular, we have

$$\|f\|_{B^s_{p,\infty}} \equiv \|S_{-1} f\|_{L^p} + \sup_q 2^{qs}\|\Delta_q f\|_{L^p}. \tag{49}$$

In [KL07] the space $B^\alpha_{3,c(\mathbb{N})}$ is defined as

$$B^\alpha_{3,c(\mathbb{N})} \equiv \left\{ f \in B^\alpha_{3,\infty} : \lim_{q\nearrow\infty} 2^{\alpha q}\|\Delta_q f\|_{L^3} = 0 \right\}. \tag{50}$$

It follows that

$$B^{\alpha+\varepsilon}_{3,\infty} \subset B^\alpha_{3,c(\mathbb{N})} \subset B^\alpha_{3,\infty} \tag{51}$$

for any $\varepsilon > 0$. To see this, first note that $B^\alpha_{3,c(\mathbb{N})} \subset B^\alpha_{3,\infty}$ is explicit in the definition (50). Next note that, for any $f \in B^{\alpha+\varepsilon}_{3,\infty}$, we have

$$2^{(\alpha+\varepsilon)q}\|\Delta_q f\|_{L^3} \leqslant M \equiv \|f\|_{B^{\alpha+\epsilon}_{3,\infty}} \Longrightarrow 2^{\alpha q}\|\Delta_q f\|_{L^3} \leqslant 2^{-\varepsilon q} M \qquad \text{for all } q. \tag{52}$$

This immediately leads to $f \in B^\alpha_{3,c(\mathbb{N})}$. Therefore $B^{\alpha+\varepsilon}_{3,\infty} \subset B^\alpha_{3,c(\mathbb{N})}$.

Proof of lemma 1

First we establish the equivalence between (C1) and (C2). The proof for the equivalence between (B1) and (B2) is almost identical and is omitted.

- (C1) $\Longrightarrow$ (C2).
  The proof is similar to that of lemma 2 and will not be repeated here.
- (C2) $\Longrightarrow$ (C1).
  We have

$$\int_0^T \|f(\cdot-\boldsymbol{y})-f(\cdot)\|_{L^3}^3 \mathrm{d}t = \int_0^T \|S_{-1}(\delta_{\boldsymbol{y}} f) + \sum_0^\infty \Delta_q(\delta_{\boldsymbol{y}} f)\|_{L^3}^3 \mathrm{d}t$$

$$\approx \int_0^T \|S_{-1}(\delta_{\boldsymbol{y}} f)\|_{L^3}^3 \mathrm{d}t + \sum_0^\infty \int_0^T \|\Delta_q(\delta_{\boldsymbol{y}} f)\|_{L^3}^3 \mathrm{d}t. \tag{53}$$

  The first term is smooth and can be easily bounded by $O(|\boldsymbol{y}|^3)$. In the following we only consider the sum. Let $N \in \mathbb{N}$ to be fixed later. We have

$$\sum_0^\infty \int_0^T \|\Delta_q(\delta_{\boldsymbol{y}} f)\|_{L^3}^3 \mathrm{d}t = \sum_0^N \int_0^T \|\Delta_q(\delta_{\boldsymbol{y}} f)\|_{L^3}^3 \mathrm{d}t + \sum_{N+1}^\infty \int_0^T \|\Delta_q(\delta_{\boldsymbol{y}} f)\|_{L^3}^3 \mathrm{d}t$$

$$\equiv A + B. \tag{54}$$

  For A, we estimate

$$\|\Delta_q(\delta_{\boldsymbol{y}} f)\|_{L^3}^3 = \int_{\mathbb{R}^3} |(\psi_q(\cdot-\boldsymbol{y}) - \psi_q(\cdot)) * \Delta_q f(\cdot)|^3 \mathrm{d}\boldsymbol{x}$$

$$\leqslant \int_{\mathbb{R}^3} |\boldsymbol{y}|^3 \|\nabla\psi_q\|_{L^\infty}^3 (\Delta_q f(\cdot))^3 \mathrm{d}\boldsymbol{x}$$

$$\lesssim 2^{3(1-\alpha)q} |\boldsymbol{y}|^3 (2^{3\alpha q} \|\Delta_q f\|_{L^3}^3). \tag{55}$$

  Therefore

$$A \lesssim 2^{3(1-\alpha)N} |\boldsymbol{y}|^3. \tag{56}$$

  For B, we have

$$\sum_{N+1}^\infty \int_0^T \|\Delta_q(\delta_{\boldsymbol{y}} f)\|_{L^3}^3 \,\mathrm{d}t \leqslant 2\sum_{N+1}^\infty \int_0^T \|\Delta_q f\|_{L^3}^3 \approx \sum_{N+1}^\infty \left(2^{3\alpha q} \int_0^T \|\Delta_q f\|_{L^3}^3\right) 2^{-3\alpha q}. \tag{57}$$

  Note that we can choose N growing slightly slower than $-\log_2 |\boldsymbol{y}|$ so that both A and B scale as $o(|\boldsymbol{y}|^{3\alpha})$. This ends the proof.

Next we show that $f \in L^3([0,T], B^\alpha_{3,\infty}) \Longrightarrow f$ satisfies (B2) with regularity α. To see this, recall that

$$f \in L^3([0,T], B^\alpha_{3,\infty}) \Longleftrightarrow \int_0^T \sup_q (2^{3\alpha q} \|\Delta_q f\|_{L^3}^3) < \infty. \tag{58}$$

As (B2) reads

$$\sup_q \int_0^T 2^{3\alpha q} \|\Delta_q f\|_{L^3}^3 \mathrm{d}t < \infty, \tag{59}$$

the conclusion follows from the fact that

$$\int_0^T 2^{3\alpha q} \|\Delta_q f\|_{L^3}^3 \mathrm{d}t \leqslant \int_0^T \sup_q (2^{3\alpha q} \|\Delta_q f\|_{L^3}^3) \tag{60}$$

for any q.

Finally we prove that $f \in L^3([0,T], B^{\alpha}_{3,c(\mathbb{N})}) \Longrightarrow f$ satisfies (C2) with regularity α. To see this, note that $f \in L^3([0,T], B^{\alpha}_{3,c(\mathbb{N})})$ implies

$$\int_0^T \|f\|^3_{B^{\alpha}_{3,\infty}} \mathrm{d}t < \infty \tag{61}$$

and

$$2^{3\alpha q}\|\Delta_q f\|^3_{L^3} \longrightarrow 0 \tag{62}$$

for all $t \in [0,T]$. Since $\|f\|^3_{B^{\alpha}_{3,\infty}} \geqslant 2^{3\alpha q}\|\Delta_q f\|^3_{L^3}$ for each q and every $t \in [0,T]$, the conclusion follows from Lebesgue's dominated convergence theorem.

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