2D Incompressible Euler Equation Linearized at Shear Flows: An Introduction

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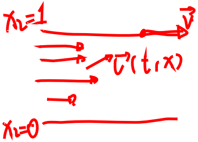
Outline

2D Incompressible Euler Equation

- 2D incompressible Euler equation in a fixed channel
  $x = (x_1, x_2) \in \Omega := \mathbb{T}_L \times (0,1)$, $\mathbb{T}_L = \mathbb{R} \setminus L\mathbb{Z}$.
  $v = (v_1, v_2)(t,x) \in \mathbb{R}^2$: velocity,
  $p = p(t,x) \in \mathbb{R}$: pressure.



$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = 0, & x \in \Omega \\ \nabla \cdot v = 0, & x \in \Omega \\ v_2(x_2 = 0, 1) = 0. \end{cases}$$

- Shear flows: $v_* = \big(U(x_2), 0\big)$, $p_* \equiv 0$.
- Question: Stability of this shear flow?

Linearized at Shear Flows

- Linearize at shear flows:

$$\begin{cases} \partial_t v_2 + U(x_2)\partial_{x_1} v_2 + \partial_{x_2} p = 0, & x \in \Omega \\ \triangle p = -2U'(x_2)\partial_{x_2} v_2, & x \in \Omega \\ \partial_{x_2} p|_{x_2=0,1} = 0, \\ v_2(x_2 = 0, 1) = 0. \end{cases}$$

- Seek an unstable solution in the form:

$$v_2(t, x_1, x_2) = e^{ik(x_1 - ct)} y(c, k, x_2) + c.c.$$

$k \in \mathbb{Z}$: wave number, $c = c_R + ic_I \in \mathbb{C}$: wave speed.
$k > 0$, $c_I > 0$

Eigenvalue Problem

- Rayleigh equation:

$$-y''(x_2) + (k^2 + \frac{U''(x_2)}{U(x_2) - c})y(x_2) = 0, \qquad x_2 \in (0,1)$$

  w/ fixed boundary condition:

$$y(0) = y(1) = 0.$$

  - $\triangleright$ $y \longleftrightarrow$ vertical component.
  - $\triangleright$ $\lambda = -ikc$: eigenvalue.
  - $\triangleright$ $(c, k > 0)$, $c_I > 0$: unstable mode.

Classical Results

When $c \notin U([\cancel{-h}, \cancel{0}])$ $[0,1)$, let ψ be $y(x_2) = \big(U(x_2) - c\big)\psi(x_2)$.

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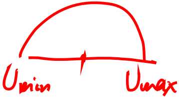
$\Longleftrightarrow$

$$\begin{cases} ((U-c)^2\psi')' - k^2(U-c)^2\psi = 0, & x_2 \in (0,1) \\ \psi(0) = \psi(1) = 0. \end{cases}$$

- Howard's semicircle theorem
- No neutral mode, i.e. $(c,k), c \in \mathbb{R}$, w/ $c \in \mathbb{R} \setminus U([0,1])$.

Classical Results



- Howard's Semicircle Theorem:

$$(c_R - \frac{U_{max} + U_{min}}{2})^2 + c_I^2 \leq (\frac{U_{max} - U_{min}}{2})^2, \quad \text{if} \quad c_I > 0.$$

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$$\int_0^1 (U-c)^2(|\psi'|^2 + k^2|\psi|^2) dx_2 = 0.$$

Let $Q := |\psi'|^2 + k^2|\psi|^2 \geq 0$.

Classical Results

$$\int_0^1 (U-c)^2 Q dx_2 = 0.$$

Imaginary:

$$\int_0^1 -2c_I(U-c_R)Q = 0 \Rightarrow \int_0^1 UQdx_2 = c_R \int_0^1 Qdx_2.$$

Real:

$$\int_0^1 ((U-c_R)^2 - c_I^2)Qdx_2 = 0 \Rightarrow \int_0^1 U^2 Qdx_2 = (c_R^2 + c_I^2) \int_0^1 Qdx_2.$$

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$$\int_0^1 \left((c_R - \tfrac{U_{max}+U_{min}}{2})^2 + c_I^2 - (\tfrac{U_{max}-U_{min}}{2})^2\right)Qdx_2 \leq 0.$$

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$y(0) = y(1) = 0 \Longrightarrow$

$$\int_0^1 |y'(x_2)|^2 + (k^2 + \frac{U''(x_2)(U(x_2)-c_R+ic_I)}{|U(x_2)-c|^2})|y(x_2)|^2 dx_2 = 0.$$

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Imaginary part:

$$c_I \int_0^1 \frac{U''(x_2)}{|U(x_2) - c|^2} |y(x_2)|^2 dx_2 = 0.$$

Seek Unstable Eigenvalues

$$\left(-\frac{d^2}{dx_2^2} + \frac{U''}{U-c}\right) y = -k^2 y.$$

$$\begin{cases} -y''(x_2) + \left(k^2 + \frac{U''(x_2)}{U(x_2)-c}\right) y(x_2) = 0, & x_2 \in (0,1) \\ y(0) = y(1) = 0. \end{cases}$$

- Treat k as a parameter.
  - No eigenvalue $\forall k \gg 1$.
- Continuation argument:
  - No eigenvalue w/ $c \in \mathbb{R} \setminus U([0,1])$
  - Semicircle theorem
  - Analyticity in $c \in \mathbb{C} \setminus U([0,1])$.

$\cos(6x_2)$

$(-\pi, \pi)$

$-1, \quad 0.$

† neutral limiting mode $(c_* \in \mathbb{R}, k_*, y_*)$: limit of unstable modes.

- Rayleigh necessary condition.

$c = U_s$: inflection value of U.

Seek Unstable Eigenvalues

Questions:

- Existence of neutral mode at $c = U_s$?
  - Class $\mathcal{K}^+$: U has exactly one inflection value U_s,

$$K(x_2) := -\frac{U''(x_2)}{U(x_2) - U_s}, \quad \text{bounded.}$$

e.g. $sin(mx_2)$, $cos(mx_2)$.

  - $-k^2$: an eigenvalue of $-\frac{d^2}{dx_2^2} - K(x_2)$.

Let

$$-k_{max}^2 = \inf_{y \in H_0^1(0,1)} \frac{\int_0^1 \left(|y'|^2 - K(x_2)|y|^2\right) dx_2}{\int_0^1 |y|^2 dx_2}$$

Seek Unstable Eigenvalues

Questions:

- Bifurcation of instability from $(c = U_s, k_{max})$?
- Linear instability when $k > k_{max}$ or $k < k_{max}$?
  - No eigenvalues for large k.
- Continuation of bifurcation curve?
  - $k \in (0, k_{max})$?

Linear Instability Arising from Inflection Value

Theorem (Lin, 2003)

If $U \in \mathcal{K}^+$, then

- *for any neutral limiting mode (c_s, k_s, y_s) with $k_s > 0$, c_s must be U_s and y_s solves system Rayleigh system with U_s, k_s,*
- $\exists \epsilon_0 < 0$ *s.t. for all $\epsilon \in (\epsilon_0, 0)$, there exists an unstable mode $(U_s + c(\epsilon), \sqrt{k_s^2 + \epsilon}, y_\epsilon)$,*
- *for all $k \in (0, k_{max})$, there is an unstable mode.*

Idea of Proof

- $c_s = U_s$.

0 $\quad$ $U''=0$ or $y=0$

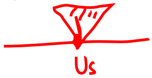
$$(-y'' + k^2 y)(\underset{}{U} - c) + U''y = 0.$$

  - $\|y_n\|_2 \le C$, where $\{y_n\}_n$: unstable solutions w/ $\{(c_n, k_n)\}_n$.
  - $y(x_{20}) \neq 0$, where $U(x_{20}) = c_s$.

- Seek unstable modes in a cone.

$y(0)=0, \quad y'(0)=1$

$y(0, c, k)$



Couette Flow $U(x_2) = x_2$

- Eigenvalue problem $\iff \begin{cases} -y''(x_2) + k^2 y(x_2) = 0, \\ y(0) = y(1) = 0. \end{cases}$

  † Only trivial solution $\Rightarrow$ spectrally stable

- $\partial_t \omega + v \cdot \omega = 0 \xrightarrow{linearize} \partial_t \omega + x_2 \partial_{x_1} \omega = 0.$

$$\begin{aligned} \omega(t, x_1, x_2) =& \omega^0(x_1 - t x_2, x_2), \\ =& \sum_{k \in \mathbb{Z}} e^{ikx_1} e^{-ikx_2 t} \omega_k^0(x_2). \end{aligned}$$

Continuous spectrum: $[0, k] \rightarrow$ inviscid damping.

Couette Flow (Linear Inviscid Damping)

Theorem (Lin-Zeng, 2011)

$\int_0^1 v_1^0 \, dx_1 = 0.$

Assume "$\int_0^1 \omega^0(x_1, x_2) dx_1 = 0$." *Let* $\omega(t, x_1, x_2)$ *be the solution of* $\partial_t \omega + x_2 \partial_{x_1} \omega = 0$ *with* $\omega(t = 0) = \omega^0(x_1, x_2)$.

1. *If* $\omega^0(x_1, x_2) \in H^{-1}_{x_1} H^1_{x_2}$, *then*

$$\|\vec{v}\|_{L^2_x} = O(\frac{1}{t}), \quad when \quad t \to \infty$$

2. *If* $\omega^0(x_1, x_2) \in H^{-1}_{x_1} H^2_{x_2}$, *then*

$$\|v_2\|_{L^2_x} = O(\frac{1}{t^2}), \quad when \quad t \to \infty.$$

Couette Flow (Linear Inviscid Damping)

$S' := \{\phi \in H^1_{x_1x_2} \mid \phi = \begin{cases} c_1 & x_2=0 \\ c_2 & x_2=1 \end{cases}\}.$

Proof. (1) $S := \{\phi \in H^1_{x_1x_2} | \phi = 0 \text{ on } \{x_2 = 0, 1\}\}$. $\nabla^\perp \phi = \Gamma$.

$\int_0^1 \partial_t v_1 + v \cdot \nabla v_1 + \partial_{x_1} p \, dx_1 = 0 \Rightarrow \partial_t \int_0^1 v_1 \, dx_1 = 0$

$$\|\vec{v}\|_{L^2_x} \leq C \sup_{\phi \in S, \|\Gamma\|_{L^2} \leq 1} \int_0^1 \int_0^1 \vec{v} \cdot \Gamma dx_2 dx_1$$

$\int_0^1 v_1^0 \, dx_1 = 0 \to \int_0^1 \omega^0 \, dx_1 = 0$

$\Rightarrow \phi = c. \quad x_2 = 0, 1$

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\le & C \sup_{\phi \in S, \|\phi\|_{H^1_0} \le 1} \int_0^1 \int_0^1 \phi \omega \, dx_2 dx_1 \\
= & C \sup_{\phi \in S, \|\phi\|_{H^1_0} \le 1} \sum_{k \ne 0} \left| \int_0^1 \omega^0_k \phi_{-k}(x_2) e^{-itkx_2} dx_2 \right|
\end{aligned}$$

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&= C \sup_{\phi \in S, \|\phi\|_{H^1_0} \le 1} \sum_{k \ne 0} \left| \int_0^1 \omega^0_k \phi_{-k}(x_2) e^{-itkx_2} dx_2 \right| \\
&= \frac{C}{t} \sup_{\|\phi\|_{H^1_0} \le 1} \sum_{k \ne 0} \left| \frac{1}{k} \int_0^1 \frac{d}{dx_2} \Big(\omega^0_k(x_2) \phi_{-k}(x_2) \Big) e^{-itkx_2} dx_2 \right|
\end{aligned}$$

Couette Flow (Linear Inviscid Damping)

$$\begin{aligned}\|\vec{v}\|_{L^2_x} \leq & \frac{C}{t} \sup_{\|\phi\|_{H^1_0}\leq 1} \left(\sum_{k\neq 0} \frac{1}{k^2} \left\|\omega^0_k(x_2)\right\|^2_{H^1_{x_2}} \right)^{1/2} \left(\sum_k \|\phi_{-k}\|^2_{H^1_{x_2}} \right)^{1/2} \\ \leq & \frac{C}{t} \left\|\omega^0\right\|_{H^{-1}_{x_1} H^1_{x_2}} .\end{aligned}$$

Couette Flow (Linear Inviscid Damping)

(2) Let ψ be the solution of $\begin{cases} -\triangle\psi = v_2, \\ \psi = 0. \quad \partial\Omega \end{cases}$

$$\|v_2\|_{L_x^2}^2 = \int_0^1 \int_0^1 \overline{v_2} v_2 dx_2 dx_1$$

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&= \int_0^1\int_0^1 \overline{\psi}\partial_{x_1}\omega^0(x_1 - tx_2, x_2) dx_2 dx_1 \\
&= \sum_{k\neq 0}\int_0^1 \overline{\psi_{-k}} ike^{-iktx_1}\omega_k^0(x_2) dx_2
\end{aligned}
$$

Couette Flow (Linear Inviscid Damping)

$$\|v_2\|_{L_x^2}^2 = \sum_{k\neq 0} \frac{1}{it^2k}\left(e^{-itkx_2}\frac{d}{dx_2}(\overline{\psi_{-k}}\omega_k^0)|_0^1 - \int_0^1 \frac{d^2}{dx_2^2}(\overline{\psi_{-k}}\omega_k^0)e^{-iktx_2}dx_2\right)$$

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$$\leq C\sum_{k\neq 0}\frac{1}{t^2|k|}\left\|\overline{\psi_k}\omega_k^0\right\|_{H_{x_2}^2} \leq C\sum_{k\neq 0}\frac{1}{t^2|k|}\left\|\psi_k\right\|_{H_{x_2}^2}\left\|\omega_k^0\right\|_{H_{x_2}^2}$$

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\leq & C\sum_{k\neq 0}\frac{1}{t^2|k|}\left\|\overline{\psi_k}\omega_k^0\right\|_{H^2_{x_2}} \leq C\sum_{k\neq 0}\frac{1}{t^2|k|}\left\|\psi_k\right\|_{H^2_{x_2}}\left\|\omega_k^0\right\|_{H^2_{x_2}}\\
\leq & \frac{C}{t^2}\sum_{k\neq 0}\frac{1}{|k|}\left\|v_{2k}\right\|_{L^2_{x_2}}\left\|\omega_k^0\right\|_{H^2_{x_2}}
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\begin{aligned}
\|v_2\|_{L^2_x}^2 =& \sum_{k\neq 0} \frac{1}{it^2 k}\left(e^{-itkx_2} \frac{d}{dx_2}(\overline{\psi_{-k}}\omega_k^0)|_0^1 - \int_0^1 \frac{d^2}{dx_2^2}(\overline{\psi_{-k}}\omega_k^0) e^{-iktx_2} dx_2 \right) \\
\leq& C \sum_{k\neq 0} \frac{1}{t^2|k|} \left\|\overline{\psi_k}\omega_k^0\right\|_{H^2_{x_2}} \leq C \sum_{k\neq 0} \frac{1}{t^2|k|} \left\|\psi_k\right\|_{H^2_{x_2}} \left\|\omega_k^0\right\|_{H^2_{x_2}} \\
\leq& \frac{C}{t^2} \sum_{k\neq 0} \frac{1}{|k|} \left\|v_{2k}\right\|_{L^2_{x_2}} \left\|\omega_k^0\right\|_{H^2_{x_2}} \\
\leq& \frac{C}{t^2} \left\|\omega^0\right\|_{H^{-1}_{x_1} H^2_{x_2}} \left\|v_2\right\|_{L^2_x}
\end{aligned}
$$

Thank you for your attention!