J Dyn Diff Equat (2009) 21:663–680
DOI 10.1007/s10884-009-9152-7

Monotone Wavefronts for Partially Degenerate Reaction-Diffusion Systems

Jian Fang · Xiao-Qiang Zhao

Received: 6 April 2009 / Published online: 21 November 2009

Abstract This paper is devoted to the study of monotone wavefronts for cooperative and partially degenerate reaction-diffusion systems. The existence of monostable wavefronts is established via the vector-valued upper and lower solutions method. It turns out that the minimal wave speed of monostable wavefronts coincides with the spreading speed. The existence of bistable wavefronts is obtained by the vanishing viscosity approach combined with the properties of spreading speeds in the monostable case.

Keywords Degenerate reaction-diffusion systems · Spreading speeds · Monostable and bistable wavefronts · Minimal wave speeds

Mathematics Subject Classification (2000) 35K57 · 35B40 · 35B20

1 Introduction

There have been extensive investigations on traveling waves and spatial dynamics for n-dimensional ($n \geq 2$) reaction-diffusion systems

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + f_i(u_1, \ldots, u_n), \quad t \geq 0,\ x \in \mathbb{R},\ 1 \leq i \leq n, \tag{1.1}$$

see, e.g., [10,14,20] and references therein. System (1.1) is a general form of various models in applied subjects such as combustion physics, chemical kinetics and spatial ecology. Usually, system (1.1) is said to be non-degenerate if each diffusion coefficient d_i is positive, and partially degenerate if some but not all diffusion coefficients are zero. There are two typical

J. Fang
Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

J. Fang · X.-Q. Zhao (✉)
Department of Mathematics and Statistics, Memorial University of Newfoundland,
St. John's, NL A1C 5S7, Canada
e-mail: zhao@mun.ca

nonlinearities for f: monostable and bistable, which depends on the number and stability of equilibria of the reaction system $\frac{du}{dt} = f(u)$. We say (1.1) is cooperative if each $f_i(u)$ is non-decreasing in all components of u with the possible exception of the ith one. In this work, we always assume that (1.1) is cooperative unless we specify it is unnecessary.

Recall that a traveling wave solution of (1.1) with speed c refers to a pair (U, c), where $U(x+ct)$ is a nontrivial and bounded solution of (1.1). We call $U = (U_1, \ldots, U_n)^T$ the wave profile and c the wave speed. We say (U, c) is a wavefront if $U(\pm\infty)$ exist and $U(-\infty) \neq U(+\infty)$.

One central problem for (1.1) with monostable nonlinearity is about the spreading speed and the minimal wave speed of wavefronts. The existence of the minimal wave speed and the stability of wavefronts for the non-degenerate case were presented by Volpert et al. [20]. The existence and estimate of the spreading speed were obtained by Weinberger et al. [23] for general cooperative reaction-diffusion systems. Li et al. [6] also showed that the spreading speed coincides with the minimal wave speed for the non-degenerate case. In population biology, there are also quite a few partially degenerate reaction-diffusion models of form (1.1). Capasso and Maddalena [1] introduced an epidemic model:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d\Delta u_1 - a_{11}u_1 + a_{12}u_2 \\ \dfrac{\partial u_2}{\partial t} = -a_{22}u_2 + g(u_1), \end{cases} \tag{1.2}$$

for which Zhao and Wang [28] established the existence of wavefronts via the scalar upper and lower solutions method. Lewis and Schmitz [11] presented the following population model:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d\Delta u_1 - \mu u_1 - \gamma_2 u_1 + \gamma_1 u_2 \\ \dfrac{\partial u_2}{\partial t} = \gamma u_2(1 - u_2/K) - \gamma_1 u_2 + \gamma_2 u_1, \end{cases} \tag{1.3}$$

for which Hadeler and Lewis [3] obtained the existence of the spreading speed, and Wang and Zhao [21] established the existence of wavefronts by reducing system (1.3) to an integral equation and then using the theory developed in [16]. Hadeler and Lewis [3] also proposed and discussed briefly the following population model:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d\Delta u_1 + f(u_1) - \gamma_1 u_1 + \gamma_2 u_2 \\ \dfrac{\partial u_2}{\partial t} = \gamma_1 u_1 - \gamma_2 u_2. \end{cases} \tag{1.4}$$

Liang et al. [7, Section 3] studied the spreading speed and periodic traveling waves of the time periodic version of (1.2) by showing that the associated solution maps are α-contractions and then using the general theory for monotone periodic semiflows. By a similar approach, Zhang and Zhao [26] established the coincidence of the spreading speed with the minimal wave speed for system (1.4). It is then natural to ask whether the spreading speed is the minimal wave speed of monotone wavefronts for general partially degenerate system (1.1) with monostable nonlinearity. The first purpose of our current paper is to give an affirmative answer to this question. Note that the solution maps associated with such a system are not compact with respect to the compact open topology. Moreover, it seems difficult to prove that these solution maps are α-contractions in the general case of nonlinearity. Thus, we may not expect to apply the abstract results developed in [22,7,8] to prove the existence of monostable wavefronts for general partially degenerate systems (1.1). We will use the vector-valued upper and lower solutions method. As always, the key point in this approach is to construct a pair of appropriate upper and lower solutions. Our construction was motivated by Weng and Zhao's work [24] on a multi-type SIS epidemic model.

In the bistable case, Volpert et al. [20] gave a complete result about the existence and stability of the unique (up to translation) wavefront when (1.1) is non-degenerate. Tsai and Sneyd [19] presented a partially degenerate buffered system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + f(u) + \sum_{i=1}^{m}[k_-^i(b_0^i - v_i) - k_+^i u v_i] \\ \dfrac{\partial v_i}{\partial t} = k_-^i(b_0^i - v_i) - k_+^i u v_i,\ 1 \leq i \leq m, \end{cases} \tag{1.5}$$

where f is of bistable type, and also studied the existence of wavefronts for (1.5) by the shooting method. Note that (1.5) can be transformed to a cooperative and bistable system on $\mathbb{R}_+ \times \prod_{i=1}^{m}[0, b_0^i]$ under the change of variable $u_i = b_0^i - v_i$, $1 \leq i \leq m$. Kazmierczak and Volpert [5] then improved the existence result in [19] by taking the limit of wave profiles of the non-degenerate systems resulted from the small perturbations of the zero diffusion coefficients. This vanishing viscosity approach was used earlier by Chen [2] for a nonlocal evolution equation and by Shen [12] for a time periodic lattice differential equation. Tsai [17] also investigated the global exponential stability of bistable wavefronts for (1.5). Xu and Zhao [25] studied the existence and global stability of wavefronts for model (1.2) in the bistable case. Recently, Jin and Zhao [4], using the shooting method, obtained the existence and global stability of bistable wavefronts for a general two-dimensional partially degenerate reaction-diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + f(u, v) \\ \dfrac{\partial v}{\partial t} = g(u, v). \end{cases} \tag{1.6}$$

Further, Tsai [18] investigated the global exponential stability of the wavefront via the squeezing method under the assumption that the partially degenerate system (1.1) admits a C^1-smooth bistable wavefront. However, the existence of bistable wavefronts for partially degenerate system (1.1) remains an open problem. The second purpose of our current paper is to solve this problem. We will utilize the vanishing viscosity method, but the technical details are quite different from those in [5,12]. In order to use this approach, we need to prove that wave speeds of the perturbed non-degenerate systems are uniformly bounded and that the limiting wave profile connects two stable equilibria. The former is done by constructing a pair of appropriate upper and lower solutions, and the latter is completed by employing the properties of spreading speeds for the monostable case.

The rest of this paper is organized as follows. In Sect. 2, we present some notations, assumptions and preliminary results. Section 3 is devoted to the construction of the required upper and lower solutions and the proof of the existence of monostable wavefronts for (1.1). It turns out the minimal wave speed coincides with the spreading speed. In Sect. 4, we establish the existence of bistable wavefronts connecting two stable equilibria for (1.1), and then make some remarks on the smoothness and global stability of these wavefronts.

2 Preliminaries

We begin with some notations. Let $\mathcal{C}$ be the set of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$. For $u = (u_1, \ldots, u_n)^T$, $v = (v_1, \ldots, v_n)^T \in \mathcal{C}$, we define $u \geq v (u \gg v)$ to mean that $u_i(x) \geq v_i(x) (u_i(x) > v_i(x))$, $1 \leq i \leq n$, $\forall x \in \mathbb{R}$, and $u > v$ to mean that $u \geq v$ but

$u \not\equiv v$. Any vector in $\mathbb{R}^n$ can be identified as an element in $\mathcal{C}$. For any $r \in \mathbb{R}$, we use boldface $\mathbf{r}$ to denote the vector with each component being r, i.e., $\mathbf{r} = (r, \dots, r)^T$. For any $\omega \gg 0$ we define $\mathcal{C}_\omega := \{u \in \mathcal{C} : \mathbf{0} \le u \le \omega\}$. We equip $\mathbb{R}^n$ with the standard norm $\| \cdot \|$.

A square matrix is said to be cooperative if all off-diagonal entries are non-negative, and irreducible if it cannot be placed into block lower-triangular form by simultaneous row/column permutations. It is easy to see that if f is differentiable and the matrix $f'(u)$ is cooperative, then (1.1) is cooperative. We denote the stability modulus of square matrix A by

$$s(A) := \max\{Re\lambda : \det(\lambda I - A) = 0\}.$$

To recall the known results for the monostable case, we need the following assumption:

(H) Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions:

(1) f is continuous with $f(\mathbf{0}) = f(\mathbf{1}) = \mathbf{0}$ and there is no ν other than $\mathbf{0}$ and $\mathbf{1}$ such that $f(\nu) = \mathbf{0}$ and $\mathbf{0} \le \nu \le \mathbf{1}$.

(2) System (1.1) is cooperative.

(3) $f(u)$ is piecewise continuously differentiable in u for $\mathbf{0} \le u \le \mathbf{1}$ and differentiable at $\mathbf{0}$, and the matrix $f'(\mathbf{0})$ is irreducible with $s(f'(\mathbf{0})) > 0$.

Let $D := diag(d_1, \dots, d_n)$. For any $\mu > 0$, define $A(\mu) := \mu^2 D + f'(\mathbf{0})$. Since $f'(\mathbf{0})$ is cooperative and irreducible, so is $A(\mu)$, $\forall \mu > 0$. Thus, $\lambda(\mu) := s(A_\mu)$ is a simple eigenvalue of $A(\mu)$ with a strongly positive eigenvector $v(\mu) = (v_1(\mu), \dots, v_n(\mu))^T$ (see, e.g., [13, Corollary 4.3.2]). We always assume $\|v(\mu)\| = 1$, $\forall \mu > 0$. Since $A(\mu) > A(0) = f'(\mathbf{0})$, we have $s(A(\mu)) > s(f'(\mathbf{0}))$ (see, e.g., [13, Corollary 4.3.2]), that is, $\lambda(\mu) > s(f'(\mathbf{0})) > 0$.

Define the function $\Phi(\mu) := \frac{\lambda(\mu)}{\mu}$, $\mu > 0$. By [8, Lemma 3.8], we then have the following properties on Φ.

Lemma 2.1 $\Phi(\mu)$ *is decreasing as* μ *near 0 and tends to infinity as* $\mu \downarrow 0$*;* $\Phi'(\mu)$ *changes sign at most once on* $(0, \infty)$ *and* $\lim_{\mu\to\infty} \Phi(\mu)$ *exists, where the limit may be infinity.*

From the above lemma, we may define $\bar{c} := \inf_{\mu>0} \Phi(\mu)$. Clearly, $\bar{c} \ge 0$.

Lemma 2.2 $\lim_{\mu\to+\infty} \Phi(\mu) = +\infty$*, and hence,* $\bar{c} > 0$.

Proof Since D is not the zero matrix, we assume, without loss of generality, that $d_1 > 0$. Let $f'(\mathbf{0}) = (a_{ij})_{n\times n}$. Since $A(\mu)v(\mu) = \lambda(\mu)v(\mu)$, we have the first component equality

$$(d_1\mu^2 + a_{11})v_1(\mu) + \sum_{k=2}^{n} a_{1k}v_k(\mu) = \lambda(\mu)v_1(\mu).$$

Note that $v(\mu) \gg \mathbf{0}$ and $a_{1k} \ge 0$, $\forall k \ge 2$. It then follows that $d_1\mu^2 + a_{11} \le \lambda(\mu)$, and hence, $\Phi(\mu) = \frac{\lambda(\mu)}{\mu} \ge d_1\mu + \frac{a_{11}}{\mu}$. Thus, $\lim_{\mu\to+\infty} \Phi(\mu) = +\infty$. Since $\Phi(\mu) > 0$, $\forall \mu > 0$ and $\lim_{\mu\downarrow 0} \Phi(\mu) = +\infty$, we then have $\bar{c} = \inf_{\mu>0} \Phi(\mu) > 0$. □

Suppose $\mu^* \in (0, +\infty)$ is the value of μ at which $\Phi(\mu)$ attains its infimum. Since $\bar{c} > 0$, we have the following result, which comes from [23, Proposition 2.1, Lemma 3.1 and Theorems 4.1, 4.2].

Lemma 2.3 *Assume* (*H*) *holds. Let* $\phi \in \mathcal{C}_{\mathbf{1}}$ *and* $u(t, x; \phi)$ *be the unique solution of (the integral form of)* (*1.1*) *through* ϕ*. Then there exists a real number* $c^* \ge \bar{c} > 0$ *such that the following statements are valid:*

(i) *If* ϕ *has compact support, then* $\lim_{t\to\infty, |x|\ge ct} u(t, x; \phi) = 0$, $\forall c > c^*$.

(ii) *For any* $c \in (0, c^*)$ *and* $r > 0$*, there is a positive number* R_r *such that for any* $\phi \in \mathcal{C}_{\mathbf{1}}$ *with* $\phi \ge \mathbf{r}$ *on an interval of length* $2R_r$*, there holds* $\lim_{t\to\infty, |x|\le ct} u(t, x; \phi) = \mathbf{1}$.

(iii) *If, in addition,* $f(\min\{\rho v(\mu^*), \mathbf{1}\}) \le \rho f'(\mathbf{0})v(\mu^*)$, $\forall \rho > 0$*, then* $c^* = \bar{c}$.

3 Monostable Wavefronts

In this section, we establish the existence of wavefronts for (1.1) in the monostable case, and further obtain the minimal wave speed and its coincidence with the spreading speed.

Throughout this section, we make the following assumption:

(K) Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions:

(1) f is continuous with $f(\mathbf{0}) = f(\mathbf{1}) = \mathbf{0}$ and there is no ν other than $\mathbf{0}$ and $\mathbf{1}$ such that $f(\nu) = \mathbf{0}$ and $\mathbf{0} \le \nu \le \mathbf{1}$.

(2) System (1.1) is cooperative.

(3) $f(u)$ is piecewise continuously differentiable in u for $\mathbf{0} \le u \le \mathbf{1}$ and differentiable at $\mathbf{0}$, and the matrix $f'(\mathbf{0})$ is irreducible with $s(f'(\mathbf{0})) > 0$.

(4) There exist $a > 0, \sigma > 1$ and $r > 0$ such that $f(u) \ge f'(\mathbf{0})u - a\|u\|^{\sigma}\mathbf{1}$ for all $\mathbf{0} \le u \le \mathbf{r}$.

(5) For any $\rho > 0$, $f(\min\{\rho v(\mu), \mathbf{1}\}) \le \rho f'(\mathbf{0})v(\mu)$, $\forall \mu \in (0, \mu^*]$, where μ^* is the value of μ at which $\Phi(\mu)$ attains its infimum.

The technical conditions (K) (4) and (5) will be used to verify lower and upper solutions for the wave profile Eq. (3.1), respectively. The condition (K)(5) implies that $f(u)$ is dominated by its linearization at $\mathbf{0}$ in the direction of $v(\mu)$. To give an example of $f(u)$ with this property, we choose $n = 2$, $D = diag\{1, d\}$ with $d < 1$, and

$$f(u_1, u_2) = (u_1[3 - 4u_1 + u_2], (1 - u_2)[u_2(3u_2 - 2) + 8u_1])^T ,$$

which is modified from [23, Example 4.1].

Let $\mathcal{B}$ be a ball in $\mathbb{R}^n$ centered at $\mathbf{0}$, $\bar{\mathcal{B}}$ the closure of $\mathcal{B}$. Then the following lemma shows that (K)(4) is automatically satisfied if f is smooth enough around the origin.

Lemma 3.1 *If there is a ball $\mathcal{B}$ such that $\frac{\partial^2 f_k(u)}{\partial u_i \partial u_j}$, $\forall 1 \le i, j, k \le n$, is continuous at every point in $\mathcal{B}$, then $(K)(4)$ holds.*

Proof By Taylor expansion, it follows that for any $u \in \mathcal{B}$,

$$f_k(u) = f_k(\mathbf{0}) + \sum_{i=1}^n \frac{\partial f_k(\mathbf{0})}{\partial u_i} u_i + \sum_{i=1}^n \sum_{j=1}^n R_{ij}(u) u_i u_j$$

with

$$|R_{ij}(u)| \le \frac{1}{2} \sup_{y \in \bar{\mathcal{B}}} \left| \frac{\partial^2 f_k(y)}{\partial u_i \partial u_j} \right| := a_{ijk}.$$

Define $a := n \max_{1 \le i,j,k \le n} a_{ijk}$. Then we have

$$\begin{aligned} f_k(u) &\ge \sum_{i=1}^n \frac{\partial f_k(\mathbf{0})}{\partial u_i} u_i - \sum_{i=1}^n \sum_{j=1}^n a_{ijk} |u_i u_j| \\ &\ge \sum_{i=1}^n \frac{\partial f_k(\mathbf{0})}{\partial u_i} u_i - \sum_{i=1}^n \sum_{j=1}^n a_{ijk} \frac{u_i^2 + u_j^2}{2} \\ &\ge \sum_{i=1}^n \frac{\partial f_k(\mathbf{0})}{\partial u_i} u_i - a \sum_{i=1}^n u_i^2. \end{aligned}$$

This implies that $f(u) \ge f'(\mathbf{0})u - a\|u\|^2\mathbf{1}$, $\forall u \in \mathcal{B}$. □

Clearly, assumption (K) ensures the conditions in Lemma 2.3. Let $c^* = \bar{c} > 0$ be the spreading speed established in Lemma 2.3.

Let $c > c^*$ be fixed. Substituting $u(t, x) \equiv U(x + ct)$ into (1.1), we get the wave profile equation

$$cU' = DU'' + f(U) \tag{3.1}$$

subject to the boundary condition

$$U(-\infty) = \mathbf{0} \quad \text{and} \quad U(+\infty) = \mathbf{1}.$$

Let $\beta > 0$ and $F(u) = (F_1(u), \dots, F_n(u))^T$ with $F_i(u) = \beta u_i + f_i(u)$. Since f is Lipschitz continuous, we can choose β sufficiently large so that $F(u) \geq F(w)$ whenever $\mathbf{1} \geq u \geq w \geq \mathbf{0}$. For a bounded solution u with $U(-\infty) = \mathbf{0}$, (3.1) is equivalent to the system

$$U_i(\xi) = G_i(U)(\xi), \quad 1 \leq i \leq n, \tag{3.2}$$

where

$$G_i(U)(\xi) = \frac{1}{\sqrt{c^2 + 4d_i\beta}} \left\{ \int_{-\infty}^{\xi} e^{\lambda_{1i}(\xi-\eta)} F_i(U(\eta)) d\eta + \int_{\xi}^{\infty} e^{\lambda_{2i}(\xi-\eta)} F_i(U(\eta)) d\eta \right\} \tag{3.3}$$

with

$$\lambda_{1i} = \begin{cases} \frac{c - \sqrt{c^2+4d_i\beta}}{2d_i}, & d_i > 0 \\ -\frac{\beta}{c}, & d_i = 0 \end{cases} \quad \text{and} \quad \lambda_{2i} = \begin{cases} \frac{c + \sqrt{c^2+4d_i\beta}}{2d_i}, & d_i > 0 \\ +\infty, & d_i = 0. \end{cases}$$

Define an operator T on $\mathcal{C}$ by

$$T(U) = (G_1(U), \dots, G_n(U))^T, \quad \forall U \in \mathcal{C}. \tag{3.4}$$

It then follows that any fixed point of T corresponds to a solution of (3.1). As in [28], we use the upper and lower fixed points of T to define the upper and lower solutions of (3.1), respectively.

Definition 3.1 A function $W \in \mathcal{C}$ is called an upper solution of (3.1) if $T(W) \leq W$. A lower solution of (3.1) is defined by reversing the inequality.

We note that if $W \in \mathcal{C}$ is a twice continuously differentiable function on $\mathbb{R}$ except finite many points ξ_i, $1 \leq i \leq m$, such that

$$DW''(\xi) - cW'(\xi) + f(W(\xi)) \leq \mathbf{0}, \quad \forall \xi \neq \xi_i,\ 1 \leq i \leq m,$$

and $W'(\xi_i+) \leq W'(\xi_i-)$, $\forall 1 \leq i \leq m$, it then easily follows that W is an upper solution of (3.1) (see, e.g., the proof of [9, Lemma 2.5]). A similar note applies to lower solutions of (3.1) if we reverse the afore-mentioned two inequalities.

The following observation is straightforward.

Lemma 3.2 *The following two statements are valid:*

(i) *T is a monotone operator on $\mathcal{C}$ in the sense that $T\phi \geq T\psi$ whenever ϕ, $\psi \in \mathcal{C}$ with $\phi \geq \psi$.*
(ii) *If ϕ is nondecreasing, then so is $T\phi$.*

We consider the function $\Phi(\mu)$ defined in section 2. By Lemma 2.1, we see that for any $c > c^*$, there exists $\mu_1 = \mu(c) \in (0, \mu^*)$ such that $\Phi(\mu_1) = c$ and $\Phi(\mu) < c, \forall \mu \in (\mu_1, \mu^*]$. For $\epsilon > 0$, let $\mu_\epsilon = \mu_1 + \epsilon$. Define $c_\epsilon := \Phi(\mu_\epsilon)$. It then follows that $c^* < c_\epsilon < c$ if ϵ is sufficiently small. Assume that $v = (v_1, \ldots, v_n)^T \gg \mathbf{0}$ and $v^\epsilon = (v_1^\epsilon, \ldots, v_n^\epsilon)^T \gg \mathbf{0}$ are the eigenvectors associated with $\lambda(\mu_1)$ and $\lambda(\mu_\epsilon)$ of $A(\mu_1)$ and $A(\mu_\epsilon)$, respectively, and $\|v\| = \|v^\epsilon\| = 1$.

Define $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_n)^T$ and $\underline{w} = (\underline{w}_1, \ldots, \underline{w}_n)^T$ with

$$\bar{w}_i = \min\{1, v_i e^{\mu_1 \xi}\} \quad \text{and} \quad \underline{w}_i = \max\{0, \gamma v_i e^{\mu_1 \xi} - v_i^\epsilon e^{\mu_\epsilon \xi}\},$$

where the positive parameters ϵ and γ will be specified later.

Let $\bar{\xi}_i = \frac{1}{\mu_1} \ln \frac{1}{v_i}$ and $\underline{\xi}_i = \frac{1}{\epsilon} \ln \frac{\gamma v_i}{v_i^\epsilon}$. Then

$$\bar{w}_i(\xi) = \begin{cases} v_i e^{\mu_1 \xi}, & \xi \le \bar{\xi}_i \\ 1, & \xi > \bar{\xi}_i \end{cases} \quad \text{and} \quad \underline{w}_i(\xi) = \begin{cases} \gamma v_i e^{\mu_1 \xi} - v_i^\epsilon e^{\mu_\epsilon \xi}, & \xi \le \underline{\xi}_i \\ 0, & \xi > \underline{\xi}_i. \end{cases}$$

Thus, it easily follows that for any $\xi \in \mathbb{R}$, $1 \le i \le n$,

$$\bar{w}_i(\xi) \le 1, \quad \bar{w}_i'(\xi+) \le \bar{w}_i'(\xi-), \quad \underline{w}_i(\xi) \le \gamma v_i e^{\mu_1 \xi} \quad \text{and} \quad \underline{w}_i'(\xi+) \ge \underline{w}_i'(\xi-).$$

Lemma 3.3 *Suppose assumption (K) holds. Then $\bar{w}$ and $\underline{w}$ are a pair of upper and lower solutions as ϵ and γ are sufficiently small.*

Proof Let a, σ, r be chosen in (K)(4). Suppose $\delta_\epsilon = c - c_\epsilon$. Choose ϵ and γ sufficiently small such that

$$\sigma \mu_1 > \mu_\epsilon, \quad c_\epsilon \in (c^*, c), \quad \gamma v \ll v^\epsilon, \quad \gamma^\sigma < \frac{\delta_\epsilon \mu_\epsilon}{a} \min_{1 \le i \le n} \{v_i^\epsilon\}, \quad \text{and} \quad \gamma v \ll \mathbf{r}.$$

Thus, $\underline{\xi}_i < 0$, $e^{\mu_\epsilon \xi} > e^{\sigma \mu_1 \xi}$, $\forall \xi \le \underline{\xi}_i$ and $\mathbf{0} < \underline{w}(\xi) \ll \mathbf{r}$, $\forall \xi \in \mathbb{R}$.

If $\xi > \bar{\xi}_i$, then $\bar{w}_i(\xi) = 1$ and

$$\begin{aligned} &d_i \bar{w}_i''(\xi) - c\bar{w}_i'(\xi) + f_i(\bar{w}(\xi)) \\ &\quad = f_i(\bar{w}_1(\xi), \ldots, \bar{w}_i(\xi), \ldots, \bar{w}_n(\xi)) \\ &\quad \le f_i(1, \ldots, \bar{w}_i(\xi), \ldots, 1) \\ &\quad = f_i(\mathbf{1}) = 0. \end{aligned}$$

If $\xi < \bar{\xi}_i$, then $\bar{w}_i(\xi) = v_i e^{\mu_1 \xi}$ and by (K)(5), we have

$$\begin{aligned} &d_i \bar{w}_i''(\xi) - c\bar{w}_i'(\xi) + f_i(\bar{w}(\xi)) \\ &\quad = v_i d_i \mu_1^2 e^{\mu_1 \xi} - v_i c \mu_1 e^{\mu_1 \xi} + f_i(\bar{w}(\xi)) \\ &\quad = \left(\mu_1^2 D v - c\mu_1 v + f'(\mathbf{0}) v\right)_i e^{\mu_1 \xi} + \left(f(\bar{w}(\xi)) - f'(\mathbf{0}) v e^{\mu_1 \xi}\right)_i \\ &\quad = (A(\mu_1) v - \lambda(\mu_1) v)_i e^{\mu_1 \xi} + \left(f(\bar{w}(\xi)) - f'(\mathbf{0}) v e^{\mu_1 \xi}\right)_i \\ &\quad = \left(f(\bar{w}(\xi)) - f'(\mathbf{0}) v e^{\mu_1 \xi}\right)_i \\ &\quad = \left(f(\min\{v e^{\mu_1 \xi}, \mathbf{1}\}) - f'(\mathbf{0}) v e^{\mu_1 \xi}\right)_i \\ &\quad \le 0. \end{aligned}$$

This indicates $\bar{w}$ is an upper solution of (3.1).

If $\xi > \underline{\xi}_i$, then $\underline{w}_i(\xi) = 0$ and

$$\begin{aligned} d_i \underline{w}_i''(\xi) - c\underline{w}_i'(\xi) + f_i(\underline{w}(\xi)) &= f_i(\underline{w}_1(\xi), \ldots, \underline{w}_i(\xi), \ldots, \underline{w}_n(\xi)) \\ &\geq f_i(0, \ldots, \underline{w}_i(\xi), \ldots, 0) \\ &= f_i(\mathbf{0}) = 0. \end{aligned}$$

If $\xi < \underline{\xi}_i$, then $\underline{w}_i(\xi) = \gamma v_i e^{\mu_1 \xi} - v_i^\epsilon e^{\mu_\epsilon \xi}$, $\underline{w}_j(\xi) \geq \gamma v_j e^{\mu_1 \xi} - v_j^\epsilon e^{\mu_\epsilon \xi}$, $\forall j \neq i$, $\|\underline{w}\|^\sigma \leq \|\gamma e^{\mu_1 \xi} v\|^\sigma = \gamma^\sigma e^{\sigma \mu_1 \xi}$ and

$$\begin{aligned} \left(f'(\mathbf{0})\underline{w}(\xi)\right)_i &= \sum_{j=1}^n \frac{\partial f_i(\mathbf{0})}{\partial u_j} \underline{w}_j(\xi) \\ &= \frac{\partial f_i(\mathbf{0})}{\partial u_i}(\gamma v_i e^{\mu_1 \xi} - v_i^\epsilon e^{\mu_\epsilon \xi}) + \sum_{j \neq i} \frac{\partial f_i(\mathbf{0})}{\partial u_j} \underline{w}_j \\ &\geq \sum_{j=1}^n \frac{\partial f_i(\mathbf{0})}{\partial u_j}(\gamma v_j e^{\mu_1 \xi} - v_j^\epsilon e^{\mu_\epsilon \xi}) \\ &= \gamma e^{\mu_1 \xi} \left(f'(\mathbf{0}) v\right)_i - e^{\mu_\epsilon \xi} \left(f'(\mathbf{0}) v^\epsilon\right)_i. \end{aligned}$$

By assumption (K)(4), we have

$$\begin{aligned} f_i(\underline{w}(\xi)) &\geq (f'(\mathbf{0})\underline{w}(\xi))_i - a\|\underline{w}\|^\sigma \\ &\geq \gamma e^{\mu_1 \xi} \left(f'(\mathbf{0}) v\right)_i - e^{\mu_\epsilon \xi} \left(f'(\mathbf{0}) v^\epsilon\right)_i - a\gamma^\sigma e^{\sigma \mu_1 \xi}. \end{aligned}$$

And hence,

$$\begin{aligned} & d_i \underline{w}_i''(\xi) - c\underline{w}_i'(\xi) + f_i(\underline{w}(\xi)) \\ &= \gamma v_i e^{\mu_1 \xi}(d_i \mu_1^2 - c\mu_1) - v_i^\epsilon e^{\mu_\epsilon \xi}(d_i \mu_\epsilon^2 - c\mu_\epsilon) + f_i(\underline{w}(\xi)) \\ &= -\gamma e^{\mu_1 \xi} \left(f'(\mathbf{0}) v\right)_i + e^{\mu_\epsilon \xi} \left(f'(\mathbf{0}) v^\epsilon\right)_i + v_i^\epsilon \mu_\epsilon \delta_\epsilon e^{\mu_\epsilon \xi} + f_i(\underline{w}(\xi)) \\ &\geq v_i^\epsilon \mu_\epsilon \delta_\epsilon e^{\mu_\epsilon \xi} - a\gamma^\sigma e^{\sigma \mu_1 \xi} \\ &\geq (v_i^\epsilon \mu_\epsilon \delta_\epsilon - a\gamma^\sigma) e^{\mu_\epsilon \xi} \\ &\geq 0. \end{aligned}$$

This suggests $\underline{w}$ is a lower solution of (1.1). □

We are now ready to prove our main result of this section, which shows that the spreading speed coincides with the minimal wave speed.

Theorem 3.1 *Assume (K) holds, and let c^* be defined as in Lemma 2.3. Then for each $c \geq c^*$, system (1.1) has a nondecreasing wavefront $U(x + ct)$ connecting $\mathbf{0}$ and $\mathbf{1}$; while for any $c \in (0, c^*)$, there is no wavefront $U(x + ct)$ connecting $\mathbf{0}$ and $\mathbf{1}$.*

Proof Let $\bar{w}$ and $\underline{w}$ be the pair of upper and lower solutions confirmed in Lemma 3.3. Consider the iteration scheme

$$U^0 := \bar{w}, \quad U^m := TU^{m-1}, \quad \forall m \geq 1.$$

By Lemmas 3.2 and 3.3, it is easy to see that

$$\mathbf{0} \leq \underline{w} \leq \ldots \leq U^m \leq U^{m-1} \leq \ldots \leq U^0 := \bar{w}.$$

Thus, $U := \lim_{m\to\infty} U^m$ exists. Since $\bar{w}$ is nondecreasing, so are $U^m, \forall m \geq 1$, and U. By the Lebesgue dominated convergence theorem, we see that U is a fixed point of T in $\mathcal{C}$. Choose $\xi^* < \min_{1\leq i\leq n} \underline{\xi}_i$. Then $\underline{w}(\xi^*) \gg 0$. Note that $\underline{w} \leq U \leq \bar{w}$, it follows that $U(-\infty) = \mathbf{0}$ and $U(+\infty) \geq U(\xi^*) \geq \underline{w}(\xi^*) \gg \mathbf{0}$, and hence, $U(+\infty) = \mathbf{1}$ because of the uniqueness of strongly positive equilibrium between $\mathbf{0}$ and $\mathbf{1}$. This gives the existence of monostable wavefronts in the case where $c > c^*$.

In the case where $c = c^*$, we use a limiting argument. Choose the sequence $\{c_m\}_{m\geq 1} \subset (c^*, \infty)$ such that $\lim_{m\to\infty} c_m = c^*$. We have known that for each c_m, there exists a wavefront $U^m = (U_1^m, \ldots, U_n^m)^T$. By the spatial translation invariance of (1.1), $U^m(-\infty) = \mathbf{0}$ and $U^m(+\infty) = \mathbf{1}$, we may assume $U_1^m(0) = \frac{1}{2}, \forall m \geq 1$. From the expression (3.2), we see that $\{\frac{dU^m}{ds}\}_{m\geq 1}$ is uniformly bounded by a straightforward computation. Note that each U^m is between $\mathbf{0}$ and $\mathbf{1}$. By Arzela–Ascoli theorem and the standard diagonal procedure, it then follows that there is a subsequence U^{m_k}, which pointwise converges to some $U^* \in \mathcal{C}$, as $k \to \infty$. Obviously, $U^*(\xi)$ is nondecreasing and $U_1^*(0) = \frac{1}{2}$. By the Lebesgue dominated convergence theorem and (3.2), we see that U^* solves (3.2) with $c = c^*$. Since $U^*(\pm\infty)$ both are zeros of f and U^* is nondecreasing, we have $U^*(-\infty) = \mathbf{0}$ and $U^*(+\infty) = \mathbf{1}$. Thus, (U^*, c^*) is a wavefront connecting $\mathbf{0}$ and $\mathbf{1}$.

The nonexistence of monostable wavefronts with speed $c \in (0, c^*)$ is a straightforward consequence of the spreading speed (see, e.g., [8, Theorem 4.3]). □

4 Bistable Wavefronts

In this section, we establish the existence of wavefronts connecting two stable equilibria in the bistable case.

Throughout this section, we make the following assumption:

(L) Assume that $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfies the following conditions:

- (1) $f(\mathbf{0}) = f(\mathbf{1}) = f(\alpha)$ with $\mathbf{0} \ll \alpha \ll \mathbf{1}$. There is no ν other than $\mathbf{0}$, $\mathbf{1}$ and α such that $f(\nu) = \mathbf{0}$ with $\mathbf{0} \leq \nu \leq 1$.
- (2) System (1.1) is cooperative.
- (3) $u \equiv \mathbf{0}$ and $u \equiv \mathbf{1}$ are unstable, and $u \equiv \alpha$ is stable, that is,
$$\lambda_0 := s(f'(\mathbf{0})) < 0, \quad \lambda_1 := s(f'(\mathbf{1})) < 0, \quad \lambda_\alpha := s(f'(\alpha)) > 0.$$
- (4) $f'(\mathbf{0})$, $f'(\mathbf{1})$ and $f'(\alpha)$ are irreducible.

Since (1.1) is cooperative, for any $b > 0$ we can choose $\beta > 0$ sufficiently large such that $Q^\beta(u) := \beta u + f(u)$ is nondecreasing in u for $-\mathbf{b} \leq u \leq \mathbf{b}$. For $d = 0$, we define a family of mappings $T_d^\beta(t) : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$, $t \geq 0$, by

$$(T_d^\beta(t)\phi)(x) = e^{-\beta t}\phi(x), \quad \forall x \in \mathbb{R}, \phi \in L^\infty(\mathbb{R});$$

and for $d > 0$, we define $T_d^\beta(t) : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ by $T_d^\beta(0) = I$ and

$$(T_d^\beta(t)\phi)(x) = e^{-\beta t} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi dt}} e^{-\frac{y^2}{4dt}} \phi(x-y)dy, \quad \forall x \in \mathbb{R}, \phi \in L^\infty(\mathbb{R}),$$

for any $t > 0$. Recall that $D = diag(d_1, \ldots, d_n)$. We further define

$$T^\beta(t) := diag(T_{d_1}^\beta(t), \ldots, T_{d_n}^\beta(t)). \tag{4.1}$$

Definition 4.1 A function $w : \mathbb{R}_+ \times \mathbb{R} \to [-\mathbf{b}, \mathbf{b}]$, where $[-\mathbf{b}, \mathbf{b}] := \{u \in \mathbb{R}^n : -\mathbf{b} \le u \le \mathbf{b}\}$, is called an upper (lower) solution of (1.1) if it satisfies

$$w(t,x) \ge (\le) T^\beta(t) w(0,x) + \int_0^t T^\beta(t-s) Q^\beta(w(s,x)) ds, \quad \forall t \ge 0, x \in \mathbb{R}. \tag{4.2}$$

Let $w = (w_1, \ldots, w_n) : \mathbb{R}_+ \times \mathbb{R} \to [-\mathbf{b}, \mathbf{b}]$ be a function with the property that w_i is C^1 in $t \ge 0$ and C^2 in $x \in \mathbb{R}$ if $d_i \ne 0$, and w_i is C^1 in $t \ge 0$ if $d_i = 0$. It is easy to see that if w satisfies

$$\frac{\partial w}{\partial t} \ge (\le) D \triangle w + f(w), \quad \forall t \ge 0, x \in \mathbb{R},$$

then w is an upper (a lower) solution of (1.1).

According to Definition 4.1, upper and lower solutions of (1.1) are not necessarily continuous. In order to prove the existence of bistable wavefronts (see the proof of Lemma 4.5), we need the following generalized comparison principle.

Lemma 4.1 *Assume $\psi^1(t,x)$ and $\psi^2(t,x)$ are a pair of lower and upper solutions of (1.1) in the sense of Definition 4.1. If $\psi^1(0,x) \le \psi^2(0,x)$, $\forall x \in \mathbb{R}$, then $\psi^1(t,x) \le \psi^2(t,x)$, $\forall t \ge 0$, $x \in \mathbb{R}$.*

Proof Let $(\phi_1, \ldots, \phi_n)^T = \phi := \psi^1 - \psi^2$ and $g_k(\theta, t, x) := Q_k^\beta(\psi^2(t,x) + \theta\phi(t,x))$, $1 \le k \le n$. Then we have

$$g_k(0,t,x) = Q_k^\beta(\psi^2(t,x)) \quad \text{and} \quad g_k(1,t,x) = Q_k^\beta(\psi^1(t,x)).$$

For any $r \in \mathbb{R}$, define $[r]_+ = \max\{r, 0\}$. Define

$$l_k := \max_{-\mathbf{b} \le u \le \mathbf{b}, 1 \le i \le n} \frac{\partial Q_k^\beta(u)}{\partial u_i}.$$

Since $Q^\beta(u)$ is nondecreasing in u for $-\mathbf{b} \le u \le \mathbf{b}$, it follows that for each $1 \le k \le n$,

$$\begin{aligned}
\phi_k(t,x) &\le \int_0^t T_{d_k}^\beta(t-s) \left[Q_k^\beta(\psi^1(s,x)) - Q_k^\beta(\psi^2(s,x)) \right] ds \\
&= \int_0^t T_{d_k}^\beta(t-s) \left(\int_0^1 \frac{d}{d\theta} g_k(\theta, s, x) d\theta \right) ds \\
&= \int_0^t T_{d_k}^\beta(t-s) \left(\sum_{i=1}^n \phi_i(s,x) \int_0^1 \frac{\partial}{\partial u_i} Q_k^\beta(\psi^2(s,x) + \theta\phi(s,x)) d\theta \right) ds \\
&\le \int_0^t T_{d_k}^\beta(t-s) \left(\sum_{i=1}^n [\phi_i(s,x)]_+ \int_0^1 \frac{\partial}{\partial u_i} Q_k^\beta(\psi^2(s,x) + \theta\phi(s,x)) d\theta \right) ds \\
&\le \int_0^t T_{d_k}^\beta(t-s) \left(l_k \sum_{i=1}^n [\phi_i(s,x)]_+ \right) ds, \ \forall t \ge 0, x \in \mathbb{R}.
\end{aligned} \tag{4.3}$$

Since the right hand side of (4.3) is nonnegative, we further have

$$[\phi_k(t,x)]_+ \le \int_0^t T_{d_k}^{\beta}(t-s)\left(l_k \sum_{i=1}^{n}[\phi_i(s,x)]_+\right)ds,\ \forall t\ge 0, x\in\mathbb{R}. \tag{4.4}$$

Let $\varpi(t,x) := \sum_{k=1}^{n}[\phi_k(t,x)]_+$ and

$$J_k(t,x) := \begin{cases} \frac{e^{-\beta t - \frac{x^2}{4d_k t}}}{\sqrt{4\pi d_k t}}, & \text{if } d_k > 0 \\ e^{-\beta t}\delta(x), & \text{if } d_k = 0, \end{cases}$$

where $\delta(x)$ is the Dirac function. By inequality (4.4), it then follows that

$$\begin{aligned} \varpi(t,x) = \sum_{k=1}^{n}[\phi_k(t,x)]_+ &\le \sum_{k=1}^{n}\int_0^t T_{d_k}^{\beta}(t-s)\left(l_k\sum_{i=1}^{n}[\phi_i(s,x)]_+\right)ds \\ &= \int_0^t \sum_{k=1}^{n}\int_{-\infty}^{\infty} l_k J_k(t-s,x-y)\varpi(s,y)dyds \\ &= \int_0^t \int_{-\infty}^{\infty} J(s,y)\varpi(t-s,x-y)dyds, \end{aligned}$$

where $J(s,y) = \sum_{k=1}^{n} l_k J_k(s,y)$. Using the same argument as in [15, Lemma 3.2], we obtain $\varpi(t,x)=0$, which implies $\psi^1(t,x)\le\psi^2(t,x)$. □

The following result is about the existence and uniqueness of bistable wavefronts for non-degenerate systems.

Lemma 4.2 ([20, Theorem 2.1]) *Consider* $\frac{\partial u}{\partial t} = A\triangle u + f(u)$, *where* $A = diag(a_1,\dots,a_n)$ *with each* $a_i > 0$. *Let the assumption (L) hold. Then there exists a unique (up to translation) monotone wavefront* $U(x+ct)$ *connecting* $\mathbf{0}$ *and* $\mathbf{1}$, *that is, a constant* c *and a twice continuously differentiable monotone vector-valued function* U, *satisfying the system* $AU'' - cU + f(U) = \mathbf{0}$ *subject to the boundary conditions* $U(-\infty) = \mathbf{0}$ *and* $U(+\infty) = \mathbf{1}$.

For any $\epsilon > 0$, let $D^{\epsilon} := diag(d_1^{\epsilon},\dots,d_n^{\epsilon})$ with

$$d_i^{\epsilon} = \begin{cases} d_i, & \text{if } d_i \ne 0 \\ \epsilon, & \text{if } d_i = 0, \end{cases}$$

and consider the following non-degenerate reaction-diffusion system

$$\frac{\partial u}{\partial t} = D^{\epsilon}\triangle u + f(u). \tag{4.5}$$

Using the same way as in Definition 4.1, we define upper and lower solutions of system (4.5). By Lemma 4.2, we immediately see that for any $\epsilon > 0$, system (4.5) admits a unique wavefront, that is, a constant c^{ϵ} and a twice continuously differentiable monotone vector-valued function U^{ϵ} with $U^{\epsilon}(-\infty) = \mathbf{0}$ and $U^{\epsilon}(+\infty) = \mathbf{1}$.

We are going to find a convergent subsequence of $\{(U^\epsilon, c^\epsilon)\}_{\epsilon\in(0,1]}$. We first prove c^ϵ, $\epsilon \in (0, 1]$, is bounded. For this purpose, we need to construct a pair of upper and lower solutions of (4.5). Choose $\rho \in C^2(\mathbb{R}, \mathbb{R})$ such that

$$\rho(\xi) = 0,\ \forall\xi \le 0;\quad \rho'(\xi) \in (0, 1),\ \forall\xi \in (0, 4);$$
$$\rho(\xi) = 1,\ \forall\xi \ge 4;\quad |\rho''(\xi)| \le 1,\ \forall\xi \in (0, 4).$$

Let $e_0, e_1 \gg 0$ with $\|e_0\| = \|e_1\| = 1$ be the eigenvectors of $f'(\mathbf{0})$ and $f'(\mathbf{1})$ associated with λ_0 and λ_1, respectively.

Lemma 4.3 *Define*

$$w_-(x - Ct; \delta, \sigma) := (\mathbf{1} + \delta e_0 - \delta e_1)\rho(\sigma(x - Ct)) - \delta e_0$$

and

$$w_+(x + Ct; \delta, \sigma) := (\mathbf{1} + \delta e_1 - \delta e_0)\rho(\sigma(x + Ct)) + \delta e_0.$$

Then there exist $\bar\delta > 0$, $\bar\sigma > 0$ and $\bar C > 0$ such that for any $\delta \in [\bar\delta/2, \bar\delta]$, $\sigma \in [\bar\sigma/2, \bar\sigma]$ and $C \ge \bar C$, $w_-(x - Ct; \delta, \sigma)$ and $w_+(x + Ct; \delta, \sigma)$ are a pair of lower and upper solutions of system (4.5) with $\epsilon \in (0, 1]$.

Proof We only prove $w_-(x-Ct; \delta, \sigma)$ is a lower solution since the proof for $w_+(x+Ct; \delta, \sigma)$ is similar. Because $\lambda_0 := s(f'(\mathbf{0})) < 0$ and $f(\delta e_0) = f'(\mathbf{0})\delta e_0 + o(\delta)e_0$ as δ is near zero, we can find $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$,

$$f(-\delta e_0) \ge -f'(\mathbf{0})\delta e_0 + \frac{1}{2}\lambda_0\delta e_0 = -\frac{1}{2}\lambda_0\delta e_0 \gg \mathbf{0}.$$

Similarly, we can find $\delta_1 > 0$ such that for any $\delta \in (0, \delta_1]$,

$$f(\mathbf{1} - \delta e_1) \ge -\frac{1}{2}\lambda_1\delta e_1 \gg \mathbf{0},$$

where $\lambda_1 := s(f'(\mathbf{1})) < 0$. Let $\bar\delta := \min\{\delta_0, \delta_1\}$. Without loss of generality, we assume that $\bar\delta < 1$ and $\mathbf{1} + \delta e_0 - \delta e_1 \gg \mathbf{0}$, $\forall\delta \in [0, \bar\delta]$. Note that $f(-\delta e_0) \gg -\frac{1}{4}\lambda_0\bar\delta e_0 \gg \mathbf{0}$ and $f(\mathbf{1} - \delta e_1) \gg -\frac{1}{4}\lambda_1\bar\delta e_1 \gg \mathbf{0}$, $\forall\delta \in [\bar\delta/2, \bar\delta]$. It then follows, by the continuity of f, that there exist $\theta_0 > 0$ and $\bar\sigma > 0$ such that for any $\theta \in [0, \theta_0]$, $\sigma \in [0, \bar\sigma]$ and $\delta \in [\bar\delta/2, \bar\delta]$, the following two inequalities hold

$$f\left((\mathbf{1} + \delta e_0 - \delta e_1)\theta - \delta e_0\right) \gg D^\epsilon(\mathbf{1} + \delta e_0 - \delta e_1)\sigma^2,\ \forall\epsilon \in (0, 1] \tag{4.6}$$

and

$$f\left((\mathbf{1} + \delta e_0 - \delta e_1)(1 - \theta) - \delta e_0\right) \gg D^\epsilon(\mathbf{1} + \delta e_0 - \delta e_1)\sigma^2,\ \forall\epsilon \in (0, 1]. \tag{4.7}$$

Let $l := \min_{\rho(\xi)\in[\theta_0, 1-\theta_0]} \rho'(\xi)$. Define

$$f_{\min} := (\min_{-\mathbf{1}\le u\le\mathbf{2}} f_1(u),\ \dots,\ \min_{-\mathbf{1}\le u\le\mathbf{2}} f_n(u))^T.$$

Choose $\bar C$ sufficiently large such that for any $C \ge \bar C$, $\sigma \in [\bar\sigma/2, \bar\sigma]$ and $\delta \in [\bar\delta/2, \bar\delta]$, there holds

$$C\sigma l(\mathbf{1} + \delta e_0 - \delta e_1) \ge -f_{\min} + D^\epsilon(\mathbf{1} + \delta e_0 - \delta e_1)\sigma^2,\ \forall\epsilon \in (0, 1]. \tag{4.8}$$

Let $\xi := x - Ct$. For convenience, we use $w_-(\xi)$ to denote $w_-(\xi; \delta, \sigma)$. To show $w_-(x - Ct; \delta, \sigma)$ is a lower solution, it suffices to verify that $D^\epsilon w_-''(\xi) + Cw_-'(\xi) + f(w_-(\xi)) \ge \mathbf{0}$,

$\forall \xi \in \mathbb{R}$, since $w_- \in C^2(\mathbb{R}, \mathbb{R})$. Now we suppose $\delta \in [\bar{\delta}/2, \bar{\delta}]$, $\sigma \in [\bar{\sigma}/2, \bar{\sigma}]$ and $C \geq \bar{C}$. Thus, inequalities (4.6), (4.7) and (4.8) hold. It follows that if $\rho(\sigma\xi) \in [0, \theta_0) \bigcup (1-\theta_0, 1]$, then

$$\begin{aligned}
&D^\epsilon w_-''(\xi) + Cw_-'(\xi) + f(w_-(\xi)) \\
&\quad = D^\epsilon(\mathbf{1} + \delta e_0 - \delta e_1)\sigma^2\rho''(\sigma\xi) + C\sigma(\mathbf{1} + \delta e_0 - \delta e_1)\rho'(\sigma\xi) + f(w_-(\xi)) \\
&\quad \geq -D^\epsilon(\mathbf{1} + \delta e_0 - \delta e_1)\sigma^2 + f((\mathbf{1} + \delta e_0 - \delta e_1)\rho(\sigma\xi) - \delta e_0) \\
&\quad \geq \mathbf{0};
\end{aligned}$$

and if $\rho(\sigma\xi) \in [\theta_0, 1-\theta_0]$, then

$$\begin{aligned}
&D^\epsilon w_-''(\xi) + Cw_-'(\xi) + f(w_-(\xi)) \\
&\quad \geq -D^\epsilon(\mathbf{1} + \delta e_0 - \delta e_1)\sigma^2 + C\sigma l(\mathbf{1} + \delta e_0 - \delta e_1) + f_{\min} \\
&\quad \geq \mathbf{0}.
\end{aligned}$$

This completes the proof. □

The following lemma shows that c^ϵ is bounded for $\epsilon \in (0, 1]$.

Lemma 4.4 $\{c^\epsilon\}_{\epsilon\in(0,1]}$ *is bounded.*

Proof By Lemma 4.3, we see that there exists $\bar{C} > 0$, $\bar{\delta}$ and $\bar{\sigma}$, independent of $\epsilon \in (0, 1]$, such that $w_-(x - \bar{C}t; \bar{\delta}, \bar{\sigma})$ and $w_+(x + \bar{C}t; \bar{\delta}, \bar{\sigma})$ are a pair of lower and upper solutions of system (4.5). Since $w_-(-\infty; \bar{\delta}, \bar{\sigma}) = -\bar{\delta}e_0 \ll \mathbf{0} = U^\epsilon(-\infty)$ and $w_-(+\infty; \bar{\delta}, \bar{\sigma}) = \mathbf{1} - \bar{\delta}e_1 \ll \mathbf{1} = U^\epsilon(+\infty)$, we see that for any $\epsilon \in (0, 1]$, there exists $\xi^\epsilon \in \mathbb{R}$ such that $U^\epsilon(\cdot + \xi^\epsilon) \geq w_-(\cdot; \bar{\delta}, \bar{\sigma})$. Thus, by the comparison principle (see Lemma 4.1), we obtain $U^\epsilon(x - c^\epsilon t + \xi^\epsilon) \geq w_-(x - \bar{C}t; \bar{\delta}, \bar{\sigma})$, $\forall t \geq 0, x \in \mathbb{R}$. Thus, $U^\epsilon(\cdot + (\bar{C} - c^\epsilon)t + \xi^\epsilon) \geq w_-(\cdot; \bar{\delta}, \bar{\sigma})$, $\forall t \geq 0$, which implies that $c^\epsilon \leq \bar{C}$, $\forall \epsilon \in (0, 1]$. Otherwise, $c^\epsilon > \bar{C}$ implies $\mathbf{0} = U^\epsilon(-\infty) \gg \mathbf{0}$, a contradiction. A similar argument gives $c^\epsilon \geq -\bar{C}$, $\forall \epsilon \in (0, 1]$. □

Lemma 4.5 *Assume* (*L*) *holds. Then the following two statements are valid:*

(i) *If $U(x + ct)$ is a non-decreasing traveling wave of (1.1) with $U(-\infty) = \alpha$ and $U(+\infty) = \mathbf{1}$, then $c > 0$.*
(ii) *If $V(x + ct)$ is a non-decreasing traveling wave of (1.1) with $V(-\infty) = \mathbf{0}$ and $V(+\infty) = \alpha$, then $c < 0$.*

Proof Let $\alpha := (\alpha_1, \ldots, \alpha_n)^T$ and $B := diag(\alpha_1, \ldots, \alpha_n)$. Since $\mathbf{0} \ll \alpha \ll \mathbf{1}$, B and $I - B$ are both invertible.

(i) Define $\hat{U} := (I - B)^{-1}(U - \alpha)$ and $E(u) := (I - B)^{-1} f((I - B)u + \alpha)$. Then $E(\mathbf{0}) = E(\mathbf{1}) = \mathbf{0}$ and $E'(u)$ is cooperative. A direct computation shows that $\hat{U}(x + ct)$ is a solution of the equation

$$\partial_t v = D\triangle v + E(v), \tag{4.9}$$

that is, $(\hat{U}, c)$ is a traveling wave of (4.9) with $\hat{U}(-\infty) = \mathbf{0}$ and $\hat{U}(+\infty) = \mathbf{1}$. Since $E'(\mathbf{0}) = (I - B)^{-1} f'(\alpha)(I - B)$, we see that $E'(\mathbf{0})$ and $f'(\alpha)$ are similar and hence, they have the same eigenvalues. Thus, it's easy to find that the nonlinearity E satisfies the assumption (H) with $f = E$. Therefore, (4.9) admits a spreading speed c_1^*, which also has a lower bound $\bar{c}_1$. By Lemma 2.2 as applied to (4.9), we have $\bar{c}_1 > 0$.

Choose $\beta > 0$ sufficiently large such that $E^\beta(u) := \beta u + E(u)$ is non-decreasing in u for $\mathbf{0} \le u \le \mathbf{1}$. Then (4.9) can be written as the following integral form

$$v(t,x) = T^\beta(t)v(0,x) + \int_0^t T^\beta(t-s)E^\beta(v(s,x))ds, \tag{4.10}$$

where $T^\beta(t)$ is defined as in (4.1). It is clear that solutions of (4.9) are solutions of (4.10). Since $(\hat{U}, c)$ is a non-decreasing traveling wave of (4.9) and $\hat{U}(+\infty) = \mathbf{1}$, we can choose $\phi \in \mathcal{C}_{\mathbf{1}}$ such that $\phi \le \hat{U}$ with $\lim_{x\to+\infty}\phi(x) = \frac{\mathbf{1}}{\mathbf{2}}$. Obviously, ϕ, as an initial function, satisfies the condition in Lemma 2.3(ii). Let $\tilde{u}(t,x)$ be the solution of (4.9) with $\tilde{u}(0,\cdot) = \phi$. By Lemma 4.1, we then have $\tilde{u}(t,x) \le \hat{U}(x+ct)$. By Lemma 2.3(ii) as applied to (4.9), we see that $\lim_{t\to\infty,|x|\le\hat{c}t}\tilde{u}(t,x) = \mathbf{1}$, $\forall \hat{c} \in (0, c_1^*)$. In particular, we have that for any $\hat{c} \in (0, \bar{c}_1)$,

$$\mathbf{1} = \lim_{t\to\infty}\tilde{u}(t,-\hat{c}t) \le \liminf_{t\to\infty}\hat{U}(-\hat{c}t+ct) = \liminf_{t\to\infty}\hat{U}\left((c-\hat{c})t\right),$$

which implies $c \ge \hat{c} > 0$ because of $\hat{U}(-\infty) = \mathbf{0}$.

(ii) Define $\hat{V} := B^{-1}(\alpha - V)$ and $J(v) = -B^{-1}f(\alpha - Bv)$. Then $J(\mathbf{0}) = J(\mathbf{1}) = \mathbf{0}$ and $J'(u)$ is cooperative. Since $J'(\mathbf{0}) = B^{-1}f'(\alpha)B$, we see that $J'(\mathbf{0})$ and $f'(\alpha)$ are similar and hence, they have the same eigenvalues. Thus, by the same argument as in (i), we obtain $c < 0$. □

Now we are ready to prove the main result of this section.

Theorem 4.1 *Assume (L) hold. Then* (1.1) *admits a monotone wavefront* (U, c) *with* $U(-\infty) = \mathbf{0}$ *and* $U(+\infty) = \mathbf{1}$.

Proof Since $\{c^\epsilon\}_{\epsilon\in(0,1]}$ is bounded, we can choose a subsequence of $\{c^{\epsilon_j}\}_{j\ge1}$, namely $\{c^j\}_{j\ge1}$, such that $\epsilon_j \to 0$ and c^j converges to some real number c. Let $D^j := diag(d_1^j, \ldots, d_n^j)$ denote D^{ϵ_j} and U^j be the corresponding wave profile of wave speed c^j. Note that $U^j(-\infty) = \mathbf{0}$ and $U^j(+\infty) = \mathbf{1}$. It then follows that there exists $\xi^j, \eta^j \in \mathbb{R}$ such that $U_1^j(\xi^j) = \alpha_1/2$ and $U_1^j(\eta^j) = (1+\alpha_1)/2$. Define

$$V^j(\cdot) := U^j(\cdot + \xi^j), \quad W^j(\cdot) := U^j(\cdot + \eta^j), \ \forall j \ge 1.$$

Then $V_1^j(0) = \alpha_1/2$ and $W_1^j(0) = (1+\alpha_1)/2$, $\forall j \ge 1$. Note that $\{V^j\}_{j\ge1}$ consists of monotone vector-valued functions. By Helly's theorem, it then follows that there exists a subsequence, still denoted by $\{V^j\}_{j\ge1}$, which converges to some monotone vector-valued function $V = (V_1, \ldots, V_n)^T$ pointwise on $\mathbb{R}$ as $j \to +\infty$. Clearly, $V_1(0) = \alpha_1/2$. Without loss of generality, we can assume $\{W^j\}_{j\ge1}$ converges to some $W = (W_1, \ldots, W_n)^T$ pointwise with $W_1(0) = (1+\alpha_1)/2$. Denote $V_\pm(\cdot) = V(\cdot \pm 0)$ and $W_\pm(\cdot) = W(\cdot \pm 0)$. Then V_-, W_- are left-continuous and V_+, W_+ are right-continuous, and $V_\pm(\xi) = V(\xi)$, $W_\pm(\xi) = W(\xi)$ almost everywhere on $\mathbb{R}$.

Next we proceed by distinguishing between two cases:

Case 1. $c \ne 0$. Choose $\beta > 0$ such that $F(u) := \beta u + f(u)$ is nondecreasing in u for $\mathbf{0} \le u \le \mathbf{1}$. We know that V^j satisfies the wave profile equation

$$D^j\frac{d^2V^j}{d\xi^2} - c^j\frac{dV^j}{d\xi} + f(V^j) = 0,$$

which is equivalent to the following integral system

$$V_i^j(\xi) = \frac{1}{\sqrt{(c^j)^2+4d_i^j\beta}} \left\{ \int_{-\infty}^{\xi} e^{\lambda_{1i}^j(\xi-\eta)} F_i(V^j(\eta))d\eta + \int_{\xi}^{\infty} e^{\lambda_{2i}^j(\xi-\eta)} F_i(V^j(\eta))d\eta \right\}, \ 1 \le i \le n \tag{4.11}$$

with

$$\lambda_{1i}^j = \frac{c^j - \sqrt{(c^j)^2 + 4d_i^j\beta}}{2d_i^j} \quad \text{and} \quad \lambda_{2i}^j = \frac{c^j + \sqrt{(c^j)^2 + 4d_i^j\beta}}{2d_i^j}.$$

By a direct computation, we have

$$\lambda_{1i} := \lim_{j\to\infty} \lambda_{1i}^j = \lim_{j\to\infty} \frac{c^j - \sqrt{(c^j)^2 + 4d_i^j\beta}}{2d_i^j} = \lim_{j\to\infty} \frac{-2\beta}{c^j + \sqrt{(c^j)^2 + 4d_i^j\beta}}$$

$$= \begin{cases} \frac{c-\sqrt{c^2+4d_i\beta}}{2d_i}, & \text{if } d_i \neq 0 \\ -\infty, & \text{if } d_i = 0, c < 0 \\ -\frac{\beta}{c}, & \text{if } d_i = 0, c > 0, \end{cases}$$

and

$$\lambda_{2i} := \lim_{j\to\infty} \lambda_{2i}^j = \begin{cases} \frac{c+\sqrt{c^2+4d_i\beta}}{2d_i}, & \text{if } d_i \neq 0 \\ -\frac{\beta}{c}, & \text{if } d_i = 0, c < 0 \\ +\infty, & \text{if } d_i = 0, c > 0. \end{cases}$$

Note that $\lim_{j\to\infty} \sqrt{(c^j)^2 + 4d_i^j\beta} = \sqrt{c^2 + 4d_i\beta} \neq 0$. By the Lebesgue dominated convergence theorem, it then follows that

$$V_i(\xi) = \frac{1}{\sqrt{c^2+4d_i\beta}} \left\{ \int_{-\infty}^{\xi} e^{\lambda_{1i}(\xi-\eta)} F_i(V(\eta))d\eta + \int_{\xi}^{\infty} e^{\lambda_{2i}(\xi-\eta)} F_i(V(\eta))d\eta \right\}, \ 1 \le i \le n, \tag{4.12}$$

which is equivalent to

$$\begin{cases} d_i V_i'' - cV_i' + f_i(V) = 0, & \text{if } d_i \neq 0 \\ -cV_i' + f_i(V) = 0, & \text{if } d_i = 0. \end{cases}$$

Thus, $V(x+ct)$ satisfies system (1.1), so does $W(x+ct)$.

Case 2. $c = 0$. By a similar argument as above, we see that

$$d_i V_i'' + f_i(V) = 0, \quad \text{if } d_i \neq 0.$$

So we only consider the case where $d_i = 0$. Since $d_i^j \frac{d^2 V_i^j}{d\xi^2} - c^j \frac{dV_i^j}{d\xi} + f_i(V^j) = 0$, for any test function $\phi \in C_0^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \left[d_i^j \frac{d^2 V_i^j(\xi)}{d\xi^2} - c^j \frac{dV_i^j(\xi)}{d\xi} + f_i(V^j(\xi)) \right] \phi(\xi) d\xi = 0,$$

which is equivalent to

$$d_i^j \int_{\mathbb{R}} V_i^j(\xi)\phi''(\xi)d\xi - c^j \int_{\mathbb{R}} V_i^j(\xi)\phi'(\xi)d\xi + \int_{\mathbb{R}} f_i(V^j(\xi))\phi(\xi)d\xi = 0. \tag{4.13}$$

Letting $j \to \infty$ in (4.13), by the Lebesgue dominated convergence theorem, we obtain $\int_{\mathbb{R}} f_i(V(\xi))\phi(\xi)d\xi = 0, \forall \phi \in C_0^\infty(\mathbb{R})$, which implies $f_i(V(\xi)) = 0$ almost everywhere on $\mathbb{R}$, and hence, $f_i(V_\pm(\xi)) = 0$ almost everywhere. Thus, for any $\xi \in \mathbb{R}$, there exists $\xi_k \to \xi - 0$ such that $f_i(V_-(\xi_k)) = 0$. Since $V_-(\xi)$ is left-continuous, we have $f_i(V_-(\xi)) = \lim_{k\to\infty} f_i(V_-(\xi_k)) = 0$. Therefore, $f_i(V_-(\xi)) = 0, \forall \xi \in \mathbb{R}$. Similarly, $f_i(V_+(\xi)) = 0 = f_i(W_\pm(\xi)), \forall \xi \in \mathbb{R}$.

Till now, we have showed that V and W satisfy the wave profile equation if $c \neq 0$, and $V_\pm$ and $W_\pm$ satisfy the wave profile equation if $c = 0$. It remains to show that the boundary conditions at $\pm\infty$ hold for V or W if $c \neq 0$ and for $V_\pm$ or $W_\pm$ if $c = 0$. Obviously $V(\pm\infty)$ both exists and are the zeros of f between $\mathbf{0}$ and $\mathbf{1}$. Since $V_1(0) = \alpha_1/2$, we have $V(-\infty) = \mathbf{0}$ and $V(+\infty) = \alpha$ or $\mathbf{1}$. Similarly, $W(+\infty) = \mathbf{1}$ and $W(-\infty) = \alpha$ or $\mathbf{0}$. By Lemma 4.5, we see that $V(+\infty) = \alpha$ and $W(-\infty) = \alpha$ cannot hold simultaneously because (V, c) and (W, c) both are non-decreasing traveling waves of (1.1) with the same wave speed. That is, either V or W is a wavefront connecting $\mathbf{0}$ and $\mathbf{1}$ if $c \neq 0$. Similarly, we see that either $V_\pm$ or $W_\pm$ is a required wavefront if $c = 0$. □

Remark 4.1 For the bistable wavefront obtained in Theorem 4.1, if $d_i > 0$, then the corresponding component of the wavefront is twice continuously differentiable; if $d_i = 0$ and $c \neq 0$, then the corresponding component of the wavefront is continuously differentiable; if $d_i = 0$ and $c = 0$, then the corresponding component of the wavefront may be discontinuous.

In order to obtain the global exponential stability with phase shift of bistable wavefronts, Tsai [18] assumed that the bistable wavefront under consideration is C^1-smooth, and that the "strong interaction" condition (see (A4) in [18]) holds. Here we have the following remark on the uniqueness and global stability of bistable wavefronts for partially degenerate system (1.1).

Remark 4.2 Assume (L) holds, and let $U(x+ct)$ be the monotone bistable wavefront obtained in Theorem 4.1. If $c \neq 0$, then $U(x + ct)$ is globally exponentially stable (with phase shift), and system (1.1) admits no other bistable wavefronts (up to translation). Indeed, by similar arguments as in the proof of [25, Theorem 3.1] and an abstract convergence theorem [27, Theorem 2.2.4], we can prove the global attractivity (with phase shift) of $U(x + ct)$, which also implies the uniqueness of bistable wavefronts. Further, a similar spectrum analysis as in [25, Section 4] proves the local exponential stability. Thus, we obtain the global exponential stability of the bistable wavefront.

As a consequence of the arguments in the proof of Theorem 4.1, we have the following remark on monostable wavefronts, which improves Theorem 3.1.

Remark 4.3 If (H) holds and $f(\min\{\rho v(\mu^*), \mathbf{1}\}) \leq \rho f'(\mathbf{0})v(\mu^*)$ for all $\rho > 0$, then the conclusion of Theorem 3.1 is still valid. Indeed, we can choose a sequence of positive numbers $\{\epsilon_j\}_{j\geq 1}$ such that $\lim_{j\to\infty} \epsilon_j = 0$, and define $D^j = D^{\epsilon_j}$. Let c_j^* be the spreading speed of (1.1) with D replaced by D^j. By Lemma 2.3 (iii), it easily follows that $\lim_{j\to\infty} c_j^* = c^* > 0$. Further, c_j^* is also the minimum wave speed of monotone wavefronts (see, e.g., [8]). For any given $c > c^*$, there is an integer $J > 0$ such that $c > c_j^* > 0$, $\forall j \geq J$. Thus, there exists a monotone wavefront $V^j(x+ct)$ of (1.1) with D replaced by D^j such that $V^j(-\infty) = \mathbf{0}$, $V^j(\infty) = \mathbf{1}$, and $V_1^j(0) = 1/2$ for all $j \geq J$. By the same limiting arguments as in the proof of Theorem 4.1, it then follows that (1.1) has a monotone wavefront $V(x+ct)$ connecting $\mathbf{0}$ to $\mathbf{1}$. Clearly, the proof of the existence of the wavefront $V(x+c^*t)$ and the nonexistence of the wavefront $V(x+ct)$ with $c \in (0, c^*)$ is the same as in Theorem 3.1.

Acknowledgments J. Fang's research is supported in part by the NSF of China (grant 10771045) and the Chinese Government Scholarship. X.-Q. Zhao's research is supported in part by the NSERC of Canada and the MITACS of Canada.

References

1. Capasso, V., Maddalena, L.: Convergence to equilibrium states for a reaction-diffusion system modelling the spatial spread of a class of bacterial and viral diseases. J. Math. Biol. **13**, 173–184 (1981)
2. Chen, X.: Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations. Adv. Differ. Equ. **2**, 125–160 (1997)
3. Hadeler, K.P., Lewis, M.A.: Spatial dynamics of the diffusive logistic equation with a sedentary compartment. Can. Appl. Math. Q. **10**, 473–499 (2002)
4. Jin, Y., Zhao, X.-Q.: Bistable waves for a class of cooperative reaction-diffusion systems. J. Biol. Dyn. **2**, 196–207 (2008)
5. Kazmierczak, B., Volpert, V.: Calcium aves in systems with immobile buffers as a limit of waves for systems with nonzero diffusion. Nonlinearity **21**, 71–96 (2008)
6. Li, B., Weinberger, H.F., Lewis, M.A.: Spreading speeds as slowest wave speeds for cooperative systems. Math. Biosci. **196**, 82–98 (2005)
7. Liang, X., Yi, Y., Zhao, X.-Q.: Spreading speeds and traveling waves for periodic evolution systems. J. Differ. Equ. **231**, 57–77 (2006)
8. Liang, X., Zhao, X.-Q.: Asymptotic speeds of spread and traveling waves for monotone semiflows with applications. Commun. Pure Appl. Math. **60**, 1–40 (Erratum: **61**(2008), 137–138)
9. Ma, S.: Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem. J. Differ Equ. **171**, 294–314 (2001)
10. Murray, J.D.: Mathematical Biology, I & II, Interdisciplinary Applied Mathematics. Springer, New York (2003)
11. Lewis, M.A., Schmitz, G.: Biological invasion of an organism with separate mobile and stationary states: Modelling and analysis. Forma **11**, 1–25 (1996)
12. Shen, W.: Traveling waves in time periodic lattice differential equations. Nonl. Anal. **54**, 319–339 (2003)
13. Smith, H.L.: Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems, Math Surveys and Monographs, vol. 41. American Mathematical Society, Providence, RI (1995)
14. Smoller, J.: Shock Waves and Reaction-Diffusion Equations, Grundlehren der mathematischen Wissenschaften **258**. Springer, New York (1994)
15. Thieme, H.R.: Asymptotic estimates of the solutions of nonlinear integral equations and asymptotic speeds for the spread of populations. J. Reine Angew. Math. **306**, 94–121 (1979)
16. Thieme, H.R., Zhao, X.-Q.: Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion models. J. Differ. Equ. **195**, 430–470 (2003)
17. Tsai, J.-C.: Asymptotic stability of traveling wave fronts in the buffered bistable system. SIAM J. Math. Anal. **39**, 138–159 (2007)
18. Tsai, J.-C.: Global exponential stability of traveling waves in monotone bistable systems. Discrete Contin. Dyn. Syst. A **21**, 601–623 (2008)

19. Tsai, J.-C., Sneyd, J.: Existence and stability of traveling waves in buffered systems. SIAM J. Appl. Math. **66**, 237–265 (2005)
20. Volpert, A.I., Volpert, V.A., Volpert, V.A.: Traveling Wave Solutions of Parabolic Systems, Translation of Mathematical Monographs, vol. 140. American Mathematical Society, Providence, RI (1994)
21. Wang, Q., Zhao, X.-Q.: Spreading speed and traveling waves for the diffusive logistic equation with a sedentary compartment. Dyn. Contin. Discrete Impuls. Syst. A **13**, 231–246 (2006)
22. Weinberger, H.F.: Long-time behavior of a class of biological models. SIAM J. Math. Anal. **13**, 353–396 (1982)
23. Weinberger, H.F., Lewis, M.A., Li, B.: Analysis of linear determinacy for spread in cooperative models. J. Math. Biol. **45**, 183–218 (2002)
24. Weng, P., Zhao, X.-Q.: Spreading speed and traveling waves for a multi-type SIS epidemic model. J. Differ. Equ. **229**, 270–296 (2006)
25. Xu, D., Zhao, X.-Q.: Bistable waves in an epidemic model. J. Dyn. Differ. Equ. **16**, 679–707 (2004)
26. Zhang, F., Zhao, X.-Q.: Asymptotic behavior of a reaction-diffusion model with a quiescent stage. Proc. R. Soc. Lond. Ser. A **463**, 1029–1043 (2007)
27. Zhao, X.-Q.: Dynamical Systems in Population Biology. Springer, New York (2003)
28. Zhao, X.-Q., Wang, W.: Fisher waves in an epidemic model. Disc. Cont. Dyn. Syst. B **4**, 1117–1128 (2004)