Wave Equations: Uniqueness and Asymptotics (Cont.)

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Proposition 1. *Let u solve*

$$u_{tt} - \triangle u = 0 \qquad \text{in } \mathbb{R}^2 \times (0, \infty) \tag{1}$$

$$u = g \quad u_t = h \qquad \text{on } \mathbb{R}^2 \times \{t = 0\} \tag{2}$$

where g, h are smooth and have compact support. Then there is a constant C such that

$$|u(x,t)| \leqslant C\,(1+t)^{-1/2}\,(1 + |t - |x||)^{-1/2}. \tag{3}$$

for all (x, t).

Proof. Assume that the supports of g, h are contained in the ball B_R. Recall the Poisson's formula:

$$u(x,t) = \frac{1}{2\,\pi\,t^2} \int_{D_t(x)} \frac{t\,g(y) + t^2\,h(y) + t\,\nabla g \cdot (y - x)}{(t^2 - |y - x|^2)^{1/2}}\,\mathrm{d}y \tag{4}$$

By taking the supreme of $g, h, \nabla g$ and noticing that $|y - x| \leqslant t$ we have

$$|u(x,t)| \leqslant C \int_{D_t(x)} \frac{\mathrm{d}y}{(t - |y - x|)^{1/2}\,(t + |y - x|)^{1/2}} \leqslant C\,t^{-1/2} \int_{D_t(x)} \frac{\mathrm{d}y}{(t - |y - x|)^{1/2}}. \tag{5}$$

Now let $z = y - x$ we have

$$|u(x,t)| \leqslant C\,t^{-1/2} \int_{D_t} \frac{\mathrm{d}z}{(t - |z|)^{1/2}}. \tag{6}$$

Here note that the integral is in fact over $D_t \cap \{|z + x| \leqslant R\}$. We have

- $|x| > t + R$: $u(x,t) \equiv 0$.
- $t - 2\,R < |x| < t + R$: We use polar coordinates, note that the angle is of order R/t (we only consider the case $t \gg R$ here), thus we have

  $$\begin{aligned} \int_{D_t \cap \{|z+x| \leqslant R\}} \frac{\mathrm{d}z}{(t - |z|)^{1/2}} &\lesssim \frac{R}{t} \int_{|x|-R}^{\min(t, |x|+R)} \frac{r\,\mathrm{d}r}{(t - r)^{1/2}} \\ &\leqslant R \int_{|x|-R}^{\min(t,|x|+R)} (t - r)^{-1/2}\,\mathrm{d}r \\ &\leqslant C\,(t - (|x| - R))^{1/2} \\ &\leqslant C\,(3\,R)^{1/2}. \end{aligned} \tag{7}$$

  since $t - 2\,R < |x| < t + R$ implies $t - 3\,R < |x| - R < t$. As

  $$-R < t - |x| < 2\,R, \tag{8}$$

  we have

  $$1 + |t - |x|| < 2\,R \implies (1 + |t - |x||)^{-1/2} > (2\,R)^{-1/2}. \tag{9}$$

  Thus

  $$\int_{D_t \cap \{|z+x| \leqslant R\}} \frac{\mathrm{d}z}{(t - |z|)^{1/2}} \leqslant C(R)\,(1 + |t - |x||)^{-1/2} \tag{10}$$

  in this case.
- $|x| < t - 2\,R$: We have $(t - |z|) \geqslant t - (R + |x|) = (t - |x|) - R \geqslant \frac{1}{2}\,(t - |x|) \geqslant \frac{R}{2\,R+1}\,(1 + t - |x|)$[1], and therefore

  $$\int_{D_t \cap \{|z+x| \leqslant R\}} \frac{\mathrm{d}z}{(t - |z|)^{1/2}} \leqslant C(R)\,(1 + t - |x|)^{-1/2}. \tag{12}$$

1. Since $|x| < t - 2\,R$, we have

$$t - |x| > 2\,R \Longrightarrow \frac{1}{t - |x|} < \frac{1}{2\,R} \Longrightarrow 1 + \frac{1}{t - |x|} < \frac{2\,R + 1}{2\,R}. \tag{11}$$

Combining the above, we see that when t is large (for example $t > 100 \max(R, 1)$), we have

$$|u(x,t)| \leqslant C t^{-1/2} (1 + |t - |x||)^{-1/2}. \tag{13}$$

On the other hand, when $t \leqslant 100 \max(R, 1)$ we have

$$\begin{aligned} |u(x,t)| &\leqslant C \int_{D_t} \frac{\mathrm{d}z}{(t^2 - |z|^2)^{1/2}} \\ &= C \int_0^t \frac{r\,\mathrm{d}r}{(t^2 - r^2)^{1/2}} \\ &= C t \leqslant C(R). \end{aligned} \tag{14}$$

Thus u is bounded by a constant when t is small and by $C\, t^{-1/2} (1 + |t - |x||)^{-1/2}$ when t is large, as a consequence, we can write

$$|u(x,t)| \leqslant C (1 + t)^{-1/2} (1 + |t - |x||)^{-1/2}. \tag{15}$$

as desired. Note that the constant C is heavily dependent on R. □

Remark 2. In general, we have

- $n > 1$ odd:

$$|u(t,x)| \leqslant C (1 + t)^{-\frac{n-1}{2}}; \tag{16}$$

- $n > 1$ even:

$$|u(t,x)| \leqslant C (1 + t)^{-\frac{n-1}{2}} (1 + |t - |x||)^{-\frac{n-1}{2}}. \tag{17}$$

Remark 3. Such algebraic decays are also characteristic in other dispersive equations, for example the Schrödinger equation.

Remark 4. It is clear that no decay can be expected for the solutions to the 1D wave equation:

$$u(x,t) = \frac{1}{2} [g(x + t) + g(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y)\, \mathrm{d}y. \tag{18}$$