MATH 527 LECTURE 18 EXERCISES AND PROBLEMS

Exercises.

Problems.

Problem 1. Prove the following theorem for 1D wave equations:

Theorem. *Assume* $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, *define* u *by*

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y. \tag{1}$$

Then

i. $u \in C^2(\mathbb{R} \times [0,\infty))$;

ii. $u_{tt} - u_{xx} = 0$ *in* $\mathbb{R} \times (0,\infty)$;

iii. u *takes the correct boundary values:*

$$\lim_{\substack{(x,t)\to(x_0,0)\\ t>0}} u(x,t) = g(x_0); \tag{2}$$

$$\lim_{\substack{(x,t)\to(x_0,0)\\ t>0}} u_t(x,t) = h(x_0). \tag{3}$$

Problem 2. (Equipartition of energy) Let $u \in C^2(\mathbb{R} \times [0,\infty))$ solve the initial-value problem for the 1D wave equation

$$u_{tt} - u_{xx} = 0, \quad u = g, \quad u_t = h; \tag{4}$$

Suppose g, h have compact support. Let

$$k(t) \equiv \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,\mathrm{d}x, \qquad p(t) \equiv \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,\mathrm{d}x \tag{5}$$

be the kinetic and potential energy. Prove that for t large enough, $k(t) = p(t) = \text{constant}$.

(Hint: Use d'Alembert's formula to compute the energy directly.)

Problem*.