Math 527 Lecture 17 Exercises and Problems

Exercises.

Problems.

Problem 1. Considerthe initial value problem in $\mathbb{R}^n \times [0, \infty)$

$$u_t - \triangle u = 0 \qquad t > 0; \qquad u = g \quad t = 0 \tag{1}$$

whose solution is given by

$$u(x, t) = \frac{1}{(4\,\pi\,t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\, g(y)\, \mathrm{d}y. \tag{2}$$

Let $g \in L^p(\mathbb{R}^n)$ with some $p \in [1, \infty]$ (including ∞!). Discuss the decay rate of $\|u\|_{L^\infty}$ as a function of t.

Problem 2. Use Fourier splitting to derive from

$$\frac{\mathrm{d}}{\mathrm{d}t} \int u^2\, \mathrm{d}x \leqslant -C \int |\nabla^m u|^2\, \mathrm{d}x. \tag{3}$$

the decay

$$\int u^2\, \mathrm{d}x \leqslant c\,(t+1)^{-\frac{n}{2m}}. \tag{4}$$

Problem 3. Consider the following equation

$$u_t - a(x, t)\, u - \nu\, \triangle u = 0 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{5}$$

where Ω is a bounded domain and $|a(x, t)| \leqslant A < \infty$ for all $x \in \Omega$. Using the energy estimate, show that there is a threshold $\nu_0 > 0$ (depending on A) such that there can be at most one solution in $C_1^2(\Omega_T)$ when $\nu \geqslant \nu_0$. (Hint: Poincaré's inequality is needed at one step of the proof.)

Problem*.