MATH 527 LECTURE 15 EXERCISES AND PROBLEMS

Exercises.

Problems.

Problem 1. Prove that if $G(x) = \frac{1}{(4\,\pi)^{-n/2}}\,e^{-|x|^2/4}$, then $\hat{G}(\xi) = e^{-|\xi|^2}$.

Problem 2. Show that, if we define the Fourier transform as that in wiki

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x}\, u(x)\,\mathrm{d}x, \tag{1}$$

we will get the same fundamental solution $\Phi(x,t)$.

Problem 3. Let u be given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\,\mathrm{d}y = \frac{1}{(4\,\pi\,t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\, g(y)\,\mathrm{d}y, \qquad t>0, \tag{2}$$

Show – by direct calculation of u_t and $\triangle u$ – that

$$u_t - \triangle u = 0 \tag{3}$$

for $t>0$.

Problem 4. Consider the equation in $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$

$$u_t - \boldsymbol{b} \cdot \nabla u = \nu\,\triangle u \tag{4}$$

where $\boldsymbol{b}$ is a constant vector. Derive its fundamental solution.

Problem*.

Problem. Consider the equation in $(x,t) \in \mathbb{R} \times \mathbb{R}^+$

$$u_t - c\,u_x = \nu\,u_{xx} \tag{5}$$

where c is a constant.

a) Derive its fundamental solution $\Phi(x,t)$.

b) Consider the initial value problem with forcing

$$u_t - c\,u_x = \nu\,u_{xx} + f(x,t) \tag{6}$$

with IV $u(x,0) = g(x)$. Prove that

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\, f(y,s)\,\mathrm{d}y\,\mathrm{d}s + \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\,\mathrm{d}y \tag{7}$$

indeed solves the problem (assume whatever regularity you need for g and f).