Heat Equation: Explicit Formulas

Nov. 4, 2011

We now turn to the heat equation

$$u_t - \triangle u = f, \quad \text{in } \Omega_T; \qquad u = g \quad \text{on } \partial^* \Omega_T \tag{1}$$

where

$$\Omega_T \equiv \Omega \times [0, T); \qquad \partial^* \Omega_T \equiv (\bar{\Omega} \times \{0\}) \cup (\partial \Omega \times [0, T]). \tag{2}$$

with $\Omega \subset \mathbb{R}^n$.

We call $\partial^* \Omega_T$ the reduced boundary of Ω_T. In this lecture we will find explicit representation formula via fundamental solution, and discuss its maximum principles.

1. Fundamental solutions and homogeneous initial-value problems.

Similar to the case of Laplace/Poisson equations, we seek a special solution in the case $\Omega = \mathbb{R}^n$ which can help representing other solutions. There are two ways finding this solution. The first one is through Fourier transform, the second one is by observing particular symmetries of the equation.

We consider the Fourier transform in the spatial variable for the initial value problem

$$u_t - \triangle u = 0, \quad t > 0; \qquad u = g, \quad t = 0. \tag{3}$$

We obtain an ODE for the function $\hat{u}(\xi, t)$:

$$(\hat{u})_t + |\xi|^2 \hat{u} = 0, \qquad \hat{u}(\xi, 0) = \hat{g}(\xi). \tag{4}$$

This equation is easy to solve:

$$\hat{u}(\xi, t) = \hat{g}(\xi) e^{-|\xi|^2 t}. \tag{5}$$

Thus all we need to do is to find the inverse Fourier transform of $e^{-|\xi|^2 t}$. From properties of the Fourier transform, if

$$\hat{G}(\xi) = e^{-|\xi|^2}, \tag{6}$$

then the Fourier transform of $t^{-n/2} G\big(x/t^{1/2}\big)$ is $e^{-|\xi|^2 t}$.

Lemma 1. *Let* $G(x) = \frac{1}{(4\pi)^{-n/2}} e^{-|x|^2/4}$, *then* $\hat{G}(\xi) = e^{-|\xi|^2}$.

Proof. Left as exercise. □

Remark 2. Note that in the above we have defined the Fourier transform as

$$\hat{u}(\xi) = \int e^{-i\xi \cdot x} u(x) \, \mathrm{d}x. \tag{7}$$

One can show that other definitions yield the same result.

Definition 3. *The function*

$$\Phi(x, t) \equiv \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & t > 0 \\ 0 & t = 0 \end{cases} \tag{8}$$

is called the fundamental solution of the heat equation.

1.1. Properties of the fundamental solution.

The fundamental solution enjoys the following properties.

1. For each time $t > 0$,

$$\int_{\mathbb{R}^n} \Phi(x, t) \, \mathrm{d}x = 1. \tag{9}$$

To see this, do a change of variable $z = \frac{x}{2\,t^{1/2}}$, which gives

$$\int_{\mathbb{R}^n} \Phi(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^n} \frac{1}{\pi^{n/2}}\, e^{-|z|^2}\,\mathrm{d}z = 1. \tag{10}$$

2. Treating t as a parameter,

$$\lim_{t \searrow 0} \Phi(x,t) = \delta \tag{11}$$

in the sense of distributions.

This follows immediately from the first property after noticing that

$$\Phi(x,t) = \varepsilon^{-n}\,\Psi\left(\frac{x}{\varepsilon}\right) \tag{12}$$

with

$$\Psi(y) = \frac{1}{(4\,\pi)^{n/2}}\, e^{-|y|^2/4}, \qquad \varepsilon = t^{1/2}. \tag{13}$$

satisfying

$$\int_{\mathbb{R}^n} \Psi(y)\,\mathrm{d}y = 1. \tag{14}$$

3. For every $t > 0$, $\Phi(x,t) \in \mathcal{S}$ the space of rapidly decreasing functions.[1]

1.2. Homogeneous initial value problem.

From the above properties we immediately obtain the explicit formula for solutions for the initial-value problem:

$$u_t - \triangle u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty); \qquad u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \tag{15}$$

Theorem 4. (Solution of initial-value problem) *Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and define u by*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\,\mathrm{d}y = \frac{1}{(4\,\pi\,t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\, g(y)\,\mathrm{d}y, \qquad t > 0, \tag{16}$$

then

i. $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$,

ii. $u_t - \triangle u = 0$ *when* $t > 0$,

iii. *u takes g as its initial value, that is*

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x\in\mathbb{R}^n, t>0}} u(x,t) = g(x_0) \tag{17}$$

for all $x_0 \in \mathbb{R}^n$.

Proof.

i. When $t > 0$, $\partial_x^\alpha \partial_t^k \Phi \in \mathcal{S}$. Thus it is clear that we can differentiate freely inside the integration:

$$\partial_x^\alpha \partial_t^k u = \int_{\mathbb{R}^n} \partial_x^\alpha \partial_t^k \Phi(x-y,t)\, g(y)\,\mathrm{d}y. \tag{18}$$

Thus $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$. Note that furthermore $u(\cdot,t) \in \mathcal{S}$ as a function of x.

ii. This is clear from how we have obtained Φ.

iii. Fix $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$. Choose $\delta > 0$ such that

$$|g(y) - g(x_0)| < \varepsilon \quad \text{if } |y - x_0| < \delta, \quad y \in \mathbb{R}^n. \tag{19}$$

1. Thus 2. and 3. combined, gives an example of how tempered distributions can be approximated by functions in $\mathcal{S}$.

Then for any $|x-x_0|<\frac{\delta}{2}$, we have

$$\begin{aligned}
|u(x,t)-g(x_0)| &= \left|\int_{\mathbb{R}^n} \Phi(x-y,t)\,[g(y)-g(x_0)]\,\mathrm{d}y\right| \\
&\leqslant \int_{B_\delta(x_0)} \Phi(x-y,t)\,|g(y)-g(x_0)|\,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n\setminus B_\delta(x_0)} \Phi(x-y,t)\,|g(y)-g(x_0)|\,\mathrm{d}y \\
&\leqslant \varepsilon \int_{B_\delta(x_0)} \Phi(x-y,t) + 2\,\|g\|_{L^\infty} \int_{\mathbb{R}^n\setminus B_\delta(x_0)} \Phi(x-y,t) \\
&\leqslant \varepsilon \int_{\mathbb{R}^n} \Phi(x-y,t) + C \int_{\mathbb{R}^n\setminus B_\delta(x_0)} t^{-n/2}\, e^{-\frac{|y-x|}{4t}}\,\mathrm{d}y \\
&= \varepsilon + C \int_{\mathbb{R}^n\setminus B_{\delta/2}(x)} t^{-n/2}\, e^{-\frac{|y-x|}{4t}}\,\mathrm{d}y. \qquad (20)
\end{aligned}$$

A change of variable shows that the latter term $\rightarrow 0$ as $t\searrow 0^+$. □

2. Nonhomogeneous problem, Duhamel's principle.

Now let us consider the nonhomogeneous case

$$u_t - \triangle u = f \quad \text{in } \mathbb{R}^n\times(0,\infty); \qquad u = g \quad \text{on } \mathbb{R}^n\times\{t=0\}. \qquad (21)$$

It is clear that we can immediately simplify the situation to the case of zero initial data:

$$u_t - \triangle u = f \quad \text{in } \mathbb{R}^n\times(0,\infty); \qquad u = 0 \quad \text{on } \mathbb{R}^n\times\{t=0\}. \qquad (22)$$

It turns out that the solution to this problem can be obtained by solving a family of homogeneous problems. We illustrate the idea in the situation of ODE systems first. Consider the ODE

$$\dot{v} - A\,v = f, \qquad v(0)=0. \qquad (23)$$

In this case we can multiply both sides of the equation by e^{-At} which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-At}\,v) = e^{-At}\,f(t), \qquad (e^{-At}\,v)(0)=0. \qquad (24)$$

and obtain the solution

$$v(t) = e^{At}\int_0^t e^{-As}\,f(s)\,\mathrm{d}s = \int_0^t e^{A(t-s)}\,f(s)\,\mathrm{d}s. \qquad (25)$$

Now consider the homogeneous system:

$$\dot{v} - A\,v = 0, \qquad v(0)=g. \qquad (26)$$

One easily sees that

$$v(t) = e^{At}\,g. \qquad (27)$$

Comparing the two results we see that the solution to the inhomogeneous equation with zero initial value can be represented as a summation of the solutions of a family of homogeneous equation with nonzero initial values:

$$v(t) = \int_0^t w(t;s)\,\mathrm{d}s \qquad (28)$$

with $w(t;\,s)$ satisfies the homogeneous equation with initial time s and initial value $f(s)$. This is the *Duhamel's principle*.

By this principle we can write down the solution:

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\,f(y,s)\,\mathrm{d}y\,\mathrm{d}s = \int_0^t \frac{1}{(4\,\pi\,(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}}\,f(y,s)\,\mathrm{d}y\,\mathrm{d}s. \qquad (29)$$

Now we need to prove

Theorem 5. (Solution of nonhomogeneous problem) *Let* $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$[2] *and have compact support. Then*

i. $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$,

ii. $u_t - \triangle u = f$ *for* $t > 0$,

iii. For each $x_0 \in \mathbb{R}^n$,

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x\in\mathbb{R}^n, t>0}} u(x,t) = 0. \tag{30}$$

Proof. First note that

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\, f(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s. \tag{31}$$

i. By assumption we can differentiate inside the integral:

$$u_t(x,t) = \int_{\mathbb{R}^n} \Phi(y,t)\, f(x-y,0)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\, f_t(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \tag{32}$$

$$\partial_{x_i x_j} u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\,(\partial_{x_i x_j} f)(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s. \tag{33}$$

ii. We calculate

$$\begin{aligned} u_t(x,t) - \triangle u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\,[(\partial_t - \triangle) f(x-y,t-s)]\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t)\, f(x-y,0)\,\mathrm{d}y \\ &\equiv A + B. \end{aligned} \tag{34}$$

– Term A.

First note that since

$$\left| \int_{\mathbb{R}^n} \Phi(y,s)\,[(\partial_t - \triangle) f(x-y,t-s)]\,\mathrm{d}y \right| \leqslant \sup\left(|(\partial_t - \triangle) f|\right) \int_{\mathbb{R}^n} \Phi(y,s)\,\mathrm{d}y \tag{35}$$

the integral is well-defined. As Φ has a singularity at $s = 0$, we write

$$\begin{aligned} A &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\,[(\partial_t - \triangle) f(x-y,t-s)]\,\mathrm{d}y\,\mathrm{d}s \\ &= \lim_{\varepsilon \searrow 0} \left[\int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\,[(-\partial_s - \triangle_y) f(x-y,t-s)]\,\mathrm{d}y\,\mathrm{d}s \right] \\ &= \lim_{\varepsilon \searrow 0} \left[\int_\varepsilon^t \int_{\mathbb{R}^n} (-\partial_s - \triangle_y))\Phi(y,s)\, f(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \right. \\ &\quad \left. + \int_{\mathbb{R}^n} \Phi(y,\varepsilon)\, f(x-y,t-\varepsilon)\,\mathrm{d}y - \int_{\mathbb{R}^n} \Phi(y,t)\, f(x-y,0)\,\mathrm{d}y \right] \\ &= \lim_{\varepsilon} \int_{\mathbb{R}^n} \Phi(y,\varepsilon)\, f(x-y,t-\varepsilon)\,\mathrm{d}y - B. \end{aligned} \tag{36}$$

Therefore

$$\begin{aligned} u_t - \triangle u &= A + B \\ &= \lim_{\varepsilon} \int_{\mathbb{R}^n} \Phi(y,\varepsilon)\, f(x-y,t-\varepsilon)\,\mathrm{d}y \\ &= \lim_{\varepsilon \searrow 0} \left[\int_{\mathbb{R}^n} \Phi(y,\varepsilon)\, f(x-y,t)\,\mathrm{d}y \right. \\ &\quad \left. + \int_{\mathbb{R}^n} \Phi(y,\varepsilon)\,[f(x-y,t-\varepsilon) - f(x-y,t)]\,\mathrm{d}y \right] \\ &= f(x,t). \end{aligned} \tag{37}$$

2. $C_n^m(\mathbb{R}^n \times [0,\infty))$ *means* f *has* m *continuous derivatives in* x *and* n *continuous derivatives in* t.

iii. It is clear that

$$\sup_x |u(x,t)| \leqslant t \sup_{x,t} |f| \to 0 \tag{38}$$

as $t \searrow 0$. □

Combining the above results, we can present the formula for the solution in the general case in the whole space:

$$u_t - \triangle u = f \quad t > 0; \qquad u = g \quad t = 0. \tag{39}$$

The solution is

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, \mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\, f(y,s)\, \mathrm{d}y\, \mathrm{d}s. \tag{40}$$

3. Regularity.

Theorem 6. (Smoothing effect of heat kernel) *Suppose $u \in C_1^2(\bar{\Omega}_T)$ solves the heat equation in Ω_T, then*

$$u \in C^\infty(\Omega_T). \tag{41}$$

We introduce the typical region considered when doing parabolic regularity:

$$C(x,t;r) \equiv \{(y,s):\ |x-y| \leqslant r,\ t - r^2 \leqslant s \leqslant t\}. \tag{42}$$

It suffices to prove that if $u \in C_1^2(C(x,t;r))$ solves the equation, then $u \in C^\infty$ inside $C(x,t;r/2)$.

Proof. We sketch the ideas. For details see L. C. Evan **Partial Differential Equations**, pp. 59 – 61.

Fix (x_0,t_0). Take a "cut-off" function η which is 0 outside $C(x_0,t_0;r)$ and 1 inside $C(x_0,t_0;3\,r/4)$. Then consider $v(x,t) \equiv \eta(x,t)\, u(x,t)$. We have

$$v_t - \triangle v = \eta_t\, u - 2\, \nabla\eta \cdot \nabla u - u\, \triangle\eta. \tag{43}$$

Now we can use the explicit formula to obtain

$$u(x,t) = v(x,t) = \int_{C(x_0,t_0;r)} \Phi(x-y,t-s)\, [(\eta_s - \triangle\eta)\, u - 2\, \nabla\eta \cdot \nabla u](y,s)\, \mathrm{d}y\, \mathrm{d}s. \tag{44}$$

for all $(x,t) \in C(x_0,t_0;3\,r/4)$.

Finally notice that, $\eta_s, \triangle\eta, \nabla\eta$ vanishes inside $C(x_0,t_0;3\,r/4)$ which means

$$u(x,t) = \int_{C(x_0,t_0;r) \backslash C(x_0,t_0;3r/4)} \Phi(x-y,t-s)\, [(\eta_s - \triangle\eta)\, u - 2\, \nabla\eta \cdot \nabla u](y,s)\, \mathrm{d}y\, \mathrm{d}s. \tag{45}$$

As a consequence, for any $(x,t) \in C(x_0,t_0;r/2)$, the integrand is uniformly bounded and can be differentiated arbitrarily. □

We further have the following estimate for derivatives.

Theorem 7. *We have*

$$\max_{C(x,t;r/2)} |\partial_x^\alpha \partial_t^l u| \leqslant \frac{C(\alpha,l)}{r^{k+2l+n+2}} \int_{C(x,t;r)} |u|\, \mathrm{d}x\, \mathrm{d}t. \tag{46}$$

Proof. This follows naturally from the proof of the last theorem. See L. C. Evan **Partial Differential Equations**, pp. 61 – 62 for details. □