Math 527 Lecture 14 Exercises and Problems

Exercises.

Problems.

Problem 1. Let

$$f(x)=|x|^s, \qquad s\in\mathbb{R}. \tag{1}$$

Consider the domains

$$\Omega=B_R, \quad \Omega'=\mathbb{R}^n\backslash B_R. \tag{2}$$

For which values of p,n,s is $f\in W^{1,p}(\Omega)$ or $W^{1,p}(\Omega')$? How about $W^{k,p}(\Omega)$, $W^{k,p}(\Omega')$?

Problem 2. Prove that any minimizer of the functional

$$I(u)=\int_\Omega |\nabla u|^2+\int f u. \tag{3}$$

is a weak solution for

$$-\triangle u=f \text{ in } \Omega, \qquad u=g \text{ on } \partial\Omega. \tag{4}$$

Then give a detailed proof of the existence of weak solutions using direct method. You can assume the existence of a function $w\in W^{1,2}(\Omega)$ whose trace on $\partial\Omega$ is exactly g.

Problem 3. Write down a proof of the higher regularity result.

Theorem. *Let $u\in W^{1,2}(\Omega)$ be a weak solution of $\triangle u=f$. If $f\in W^{k,2}(\Omega)$, then $u\in W^{k+2,2}(\Omega')$ for any $\Omega'\subset\subset\Omega$, and*

$$\|u\|_{W^{k+2,2}(\Omega')}\leqslant C\left(\|u\|_{L^2(\Omega)}+\|f\|_{W^{k,2}(\Omega)}\right). \tag{5}$$

Here the constant depends on $d,h,\operatorname{dist}(\Omega',\Omega)$.

Problem 4. Let u,v be weak solutions to the same Poisson equation

$$-\triangle u=f \text{ in } \Omega, \qquad u=g \text{ on } \partial\Omega. \tag{6}$$

Show that $w=u-v$ solves (as a weak solution)

$$-\triangle w=0 \text{ in } \Omega, \qquad u=0 \text{ on } \partial\Omega. \tag{7}$$

Then use definition of weak solution to prove $w=0$.

Problem*.

Problem. Use the L^2 version of the Calderon-Zygmund inequality:

$$\|\nabla^2 w\|_{L^2(\Omega)}\leqslant\|f\|_{L^2(\Omega)} \tag{8}$$

where w is the Newton potential of f, to prove the inner regularity estimate on a ball: for any weak solution u of $\triangle u=f$, we have

$$\|u\|_{W^{2,2}(B_r)}\leqslant C\,(\|u\|_{L^2(B_R)}+\|f\|_{L^2(B_R)}). \tag{9}$$

Where $r<R$. For simplicity, you do not need to worry about the regularity of u, that is, work as if $u\in C^\infty$.

Hint: Follow these steps.

1. For any $v\in C_0^\infty(B_R)$, $v=\int\ \Gamma(x,y)\,(\triangle v)(y)\,\mathrm{d}y$. That is, v is the Newton potential of its Laplacian.
2. Use a cut-off function η whose support is in $B_{r'}$ with $r<r'<R$, apply C-Z inequality on $v=\eta u$ to obtain
$$\|\nabla^2 u\|_{L^2(B_r)}\leqslant C\,(\|u\|_{L^2(B_{r'})}+\|f\|_{L^2(B_{r'})}+\|\nabla u\|_{L^2(B_{r'})}). \tag{10}$$
3. Use another cut-off function ξ whose support is in B_R, to obtain
$$\|\nabla u\|_{L^2(B_{r'})}\leqslant C\,(\|u\|_{L^2(B_R)}+\|f\|_{L^2(B_R)}). \tag{11}$$