Poisson Equation in Sobolev Spaces

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Today we discuss the Poisson equation

$$-\triangle u = f \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega \tag{1}$$

in Sobolev spaces. It's existence, uniqueness, and regularity.

Weak Solution.

Definition 1. *$u \in W^{1,2}(\Omega)$ is a weak solution of the Poisson equation*

$$\triangle u = f, \quad \text{in } \Omega, \qquad u = g, \quad \text{on } \partial\Omega \tag{2}$$

if

$$\int \nabla u \cdot \nabla v + \int f v = 0 \quad \forall v \in W_0^{1,2}(\Omega); \qquad u - g \in W_0^{1,2}(\Omega). \tag{3}$$

The weak formulation is advantageous in getting quick estimates. For example, when $g = 0$, we have

$$\|u\|_{W^{1,2}} \leqslant C\, \|f\|_{L^2} \tag{4}$$

for some constant C.

To see this, note that when $g = 0$, $u \in W_0^{1,2}$ can be used as a test function, which gives

$$\int |\nabla u|^2 = -\int f u \leqslant \|f\|_{L^2} \|u\|_{L^2}. \tag{5}$$

Applying Poincaré inequality gives the desired estimate.

Existence.

The direct method shows the existence/uniqueness of the solution of PDEs by studying its variational formulation. We sketch this approach by studying the Poisson equation with zero boundary condition:

$$\triangle u = f, \qquad u \in W_0^{1,2}(\Omega). \tag{6}$$

We know that any weak solution to this problem is a minimizer of the functional

$$I(u) = \int_\Omega |\nabla u|^2 + \int f u. \tag{7}$$

We would like to show that the minimizer exists. We apply the so-called "direct method": Assume $f \in L^2$.

1. Writing
$$I(u) \geqslant \int_\Omega |\nabla u|^2 - \varepsilon \int u^2 - \frac{1}{4\,\varepsilon} \int f^2 \tag{8}$$
and recalling the Poincaré's inequality, we see that $I(u)$ has finite infimum.

2. Let u_n be such that $I(u_n) \searrow \inf_{u \in W_0^{1,2}} I(u)$. We show that there is a subsequence converging to some limit $u_\infty \in W_0^{1,2}$. To see this, note that a uniform bound on $I(u_n)$ implies a uniform bound on $\int |\nabla u_n|^2$, since
$$I(u) \geqslant \|\nabla u\|_{L^2}^2 - \|u\|_{L^2} \|f\|_{L^2} \geqslant \|\nabla u\|_{L^2}^2 - C\, \|\nabla u\|_{L^2} = (\|\nabla u\|_{L^2} - C)\, \|\nabla u\|_{L^2}. \tag{9}$$
by Hölder's inequality and Poincaré's inequality.

3. Uniform boundedness of $\|\nabla u_n\|_{L^2}$ implies that u_n is uniformly bounded in $W_0^{1,2}$ and thus has a weakly[1] converging subsequence, still denoted by u_n. We denote the limit by u_∞.

1. The weak convergence is in $W^{1,2}$. Recall that a sequence $\{u_n\}$ in a Hilbert space H is weakly convergent with weak limit $u_\infty \in H$ if $(u_n, v) \longrightarrow (u_\infty, v)$ for any $v \in H$.

Furthermore, using compact embedding, we see that when u_n converges to u_∞ weakly in $W^{1,2}$, we can find a subsequence, still denoted u_n, converging to u_∞ strongly in L^2, at the same time ∇u_n converges to ∇u_∞ weakly in L^2.

4. The convexity of the functional $I(u)$ then guarantees that

$$I(u_\infty) \leqslant \liminf_{n \nearrow \infty} I(u_n) = \inf_{u \in W_0^{1,2}} I(u) \tag{10}$$

which means u_∞ is a minimizer.

Uniqueness.

Let u, v be weak solutions to

$$-\triangle u = f \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega \tag{11}$$

Let $w = u - v$. Then w is weak solution to

$$-\triangle w = 0 \text{ in } \Omega, \qquad w = 0 \text{ on } \partial\Omega. \tag{12}$$

Note that as $w \notin C^2$, we cannot directly apply the maximum principle for harmonic functions. However proving through the definition of weak solution is as simple. Left as a problem.

Interior Regularity.

Our goal is the prove the following theorem, which justifies the intuition that u is twice more differentiable than f. By "interior regularity", we mean we do not deal with boundary data, and therefore the L^2-norm of u is necessary in the RHS.

Theorem 2. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\triangle u = f$ with $f \in L^2(\Omega)$. For any $\Omega' \subset \subset \Omega$, we have $u \in W^{2,2}(\Omega')$, and*

$$\|u\|_{W^{2,2}(\Omega')} \leqslant C\,(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}). \tag{13}$$

where the constant depends on the distance between Ω' and $\partial\Omega$. Furthermore, $\triangle u = f$ almost everywhere in Ω.

Remark 3. The difficulty in proving the theorem lies in the fact that we have to show $u \in W^{2,2}$. Once this is known, the estimate is relatively easy to establish.

1. We first show that

$$\|\nabla u\|_{L^2(\Omega')}^2 \leqslant \frac{17}{\delta^2}\|u\|_{L^2(\Omega)}^2 + \delta^2\|f\|_{L^2(\Omega)}^2. \tag{14}$$

without any extra assumptions.

Let $\eta(x)$ be a "cut-off" function defined by

$$\eta(x) = \begin{cases} 1 & x \in \Omega' \\ 1 - \frac{1}{\delta}\operatorname{dist}(x, \Omega') & 0 \leqslant \operatorname{dist}(x, \Omega') \leqslant \delta \\ 0 & \operatorname{dist}(x, \Omega') > \delta \end{cases} \tag{15}$$

and set the test function

$$v = \eta^2 u \in W_0^{1,2}(\Omega). \tag{16}$$

Some calculation yields

$$\int_\Omega \eta^2 |\nabla u|^2 + 2\int_\Omega (\eta \nabla u) \cdot (u \nabla \eta) = -\int_\Omega \eta^2 f u. \tag{17}$$

Using Young's inequality

$$|a\,b| \leqslant \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2 \qquad a, b \in \mathbb{R},\ \varepsilon > 0 \tag{18}$$

on the 2nd term on the LHS and on the RHS we have

$$\int_\Omega \eta^2 |\nabla u|^2 \leqslant \frac{1}{2}\int_\Omega \eta^2 |\nabla u|^2 + 2\int_\Omega u^2 |\nabla \eta|^2 + \frac{1}{2\delta^2}\int \eta^2 u^2 + \frac{\delta^2}{2}\int \eta^2 f^2. \tag{19}$$

Moving the first term on the RHS to the left, we have

$$\int_{\Omega'} |\nabla u|^2 \leqslant \left(\frac{16}{\delta^2} + \frac{1}{\delta^2}\right)\int_\Omega u^2 + \delta^2 \int_\Omega f^2. \tag{20}$$

2. Note that $\int |\triangle u|^2 = \int |\nabla^2 u|^2 +$ boundary terms. if we assume $u \in W^{3,2}$. Thus using $\triangle u$ as test function we obtain

$$\|\nabla^2 u\|_{L^2(\Omega')}^2 \leqslant \|f\|_{L^2(\Omega)}^2. \tag{21}$$

Proof. Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, with $\operatorname{dist}(\Omega'', \partial\Omega) \geqslant \delta/4$, $\operatorname{dist}(\Omega', \partial\Omega'') \geqslant \delta/4$.

Now choose $\eta \in C_0^1(\Omega'')$ with $\eta = 1$ on Ω' and $|\nabla \eta| \leqslant 8/\delta$, and set

$$v = \eta^2 \triangle_i^h u \tag{22}$$

where

$$\triangle_i^h u(x) = \frac{u(x + h\, e_i) - u(x)}{h} \tag{23}$$

Then we have

$$\begin{aligned} \int_{\Omega''} \nabla(\triangle_i^h u) \cdot \nabla v &= \int_{\Omega''} \triangle_i^h (\nabla u) \cdot \nabla v \\ &= -\int_{\Omega''} \nabla u \cdot \nabla(\triangle_i^h v) \\ &= \int_{\Omega''} f\, \triangle_i^h v \\ &\leqslant \|f\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega'')}. \end{aligned} \tag{24}$$

Recalling

$$v = \eta^2 \triangle_i^h u \tag{25}$$

We have

$$\int_{\Omega''} \nabla(\triangle_i^h u) \cdot \nabla(\eta^2 \triangle_i^h u) \leqslant \|f\|_{L^2(\Omega)} \|\nabla(\eta^2 \triangle_i^h u)\|_{L^2(\Omega'')}. \tag{26}$$

The terms can be expanded to obtain

$$\begin{aligned} \int_{\Omega''} \eta^2 |\nabla(\triangle_i^h u)|^2 &\leqslant \|f\|_{L^2(\Omega)} \|\nabla(\eta^2 \triangle_i^h u)\|_{L^2(\Omega'')} - 2\int_{\Omega''} (\eta\, \nabla \triangle_i^h u) \cdot (\triangle_i^h u\, \nabla \eta) \\ &\leqslant 2\|f\|_{L^2(\Omega)}^2 + \frac{1}{8} \|\nabla(\eta^2 \triangle_i^h u)\|_{L^2(\Omega'')}^2 + \frac{1}{4} \int_{\Omega''} |\eta\, \nabla \triangle_i^h u|^2 + 8 \int_{\Omega''} |\nabla \eta|^2 |\triangle_i^h u|^2. \end{aligned} \tag{27}$$

This gives

$$\frac{3}{4} \int_{\Omega''} |\eta\, \nabla \triangle_i^h u|^2 \leqslant 2\|f\|_{L^2(\Omega)}^2 + \frac{1}{8} \|\nabla(\eta^2 \triangle_i^h u)\|_{L^2(\Omega'')}^2 + 8 \int |\nabla \eta|^2 |\triangle_i^h u|^2. \tag{28}$$

To proceed further, we need to study the two terms $\frac{1}{8} \|\nabla(\eta^2 \triangle_i^h u)\|_{L^2(\Omega'')}^2$ and $8 \int |\nabla \eta|^2 |\triangle_i^h u|^2$.

– $\frac{1}{8} \|\nabla(\eta^2 \triangle_i^h u)\|_{L^2(\Omega'')}^2$. We have

$$\begin{aligned} \|\nabla(\eta^2 \triangle_i^h u)\|_{L^2(\Omega'')}^2 &\leqslant 2\|(\nabla(\eta^2))\, |\triangle_i^h u|\|_{L^2}^2 + 2\|\eta^2 |\nabla \triangle_i^h u|\|_{L^2}^2 \\ &\leqslant 2(\sup |\nabla(\eta^2)|)\, \|\triangle_i^h u\|_{L^2}^2 + 2\|\eta |\nabla \triangle_i^h u|\|_{L^2}^2. \end{aligned} \tag{29}$$

Where we have used the fact that $\eta^2 \leqslant \eta$.

– $8 \int |\nabla \eta|^2 |\triangle_i^h u|^2$. We have

$$\int |\nabla \eta|^2 |\triangle_i^h u|^2 \leqslant (\sup |\nabla \eta|^2) \int |\triangle_i^h u|^2 = (\sup |\nabla \eta|^2)\, \|\triangle_i^h u\|_{L^2}^2. \tag{30}$$

Thus we have

$$\frac{1}{2} \int_{\Omega''} |\eta\, \nabla \triangle_i^h u|^2 \leqslant 2\|f\|_{L^2(\Omega)}^2 + \frac{1}{4} (\sup |\nabla(\eta^2)|)\, \|\triangle_i^h u\|_{L^2}^2 + 8(\sup |\nabla \eta|^2)\, \|\triangle_i^h u\|_{L^2}^2. \tag{31}$$

The following lemma then guarantees the existence of $\nabla^2 u$ and also gives the desired estimate.

Lemma. *Let*

$$\triangle_i^h u \equiv \frac{u(x + h\, e_i) - u(x)}{h}, \qquad h \neq 0 \tag{32}$$

with e_i being the ith unit vector of $\mathbb{R}^n$. Let $\Omega' \subset\subset \Omega$ and $|h| < \operatorname{dist}(\Omega', \partial\Omega)$. Then

1. *If $u \in L^2(\Omega)$ and there is $K < \infty$ such that*
$$\|\triangle_i^h u\|_{L^2(\Omega')} \leqslant K \tag{33}$$
then $u \in W^{1,2}(\Omega')$ and
$$\|\partial_{x_i} u\|_{L^2(\Omega')} \leqslant K. \tag{34}$$

2. *Conversely, if $u \in W^{1,2}(\Omega')$, then $\triangle_i^h u \in L^2(\Omega')$ with*
$$\|\triangle_i^h u\|_{L^2(\Omega')} \leqslant \|\partial_{x_i} u\|_{L^2(\Omega')}. \tag{35}$$

Proof.

1. We first show that $\triangle_i^h u$ converges as distributions in $\mathcal{D}'(\Omega')$ to the distributional derivative of u. Check
$$\int_{\Omega'} (\triangle_i^h u)\, \varphi = -\int u\, (\triangle_i^{-h} \varphi) \longrightarrow -\int u\, (\partial_{x_i} \varphi) \tag{36}$$
by Lebesgue's dominated convergence theorem.
We have
$$(\partial_{x_i} u)(\varphi) = \lim \int_{\Omega'} (\triangle_i^h u)\, \varphi \leqslant \|\triangle_i^h u\|_{L^2(\Omega')} \|\varphi\|_{L^2} \leqslant K\, \|\varphi\|_{L^2}, \qquad \forall \varphi \in C_0^\infty(\Omega'). \tag{37}$$
Now recall that $C_0^\infty(\Omega')$ is dense in $L^2(\Omega')$, $(\partial_{x_i} u)$ can be identified with a bounded linear operator on L^2, which means it can be identified with a function in $L^2(\Omega')$.[2]

2. Since C^∞ is dense in $W^{1,2}$, we only need to consider the case when $u \in C^\infty \cap W^{1,2}$. In this case we have
$$\triangle_i^h u(x) = \frac{1}{h} \int_0^h \partial_{x_i} u(x_1, \ldots, x_{i-1}, x_i + s, x_{i+1}, \ldots, x_n)\, \mathrm{d}s. \tag{38}$$
This gives
$$|\triangle_i^h u(x)|^2 \leqslant \frac{1}{h} \int_0^h |\partial_{x_i} u(x_1, \ldots, x_{i-1}, x_i + s, x_{i+1}, \ldots, x_n)|^2\, \mathrm{d}s \tag{39}$$
due to Hölder's inequality. Now integrate over Ω' and exchange the order of integration on the RHS we obtain the result. □

With the help of this lemma (part b)) we have
$$\frac{1}{2} \int_{\Omega''} |\eta\, \nabla \triangle_i^h u|^2 \leqslant 2\, \|f\|_{L^2(\Omega)}^2 + \frac{1}{4} (\sup |\nabla(\eta^2)|)\, \|\partial_{x_i} u\|_{L^2}^2 + 8\, (\sup |\nabla \eta|^2)\, \|\partial_{x_i} u\|_{L^2}^2. \tag{40}$$
which is a uniform bound on
$$\|\triangle_i^h \nabla u\|_{L^2(\Omega'')} \leqslant \int_{\Omega''} |\eta\, \nabla \triangle_i^h u|^2. \tag{41}$$
Now part a) of the lemma yields $\partial_{x_i} \nabla u \in L^2(\Omega'')$ and also the desired estimate. □

When f has better regularity, we can differentiate the equation first and obtain the following interior regularity result.

Theorem 4. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\triangle u = f$. If $f \in W^{k,2}(\Omega)$, then $u \in W^{k+2,2}(\Omega')$ for any $\Omega' \subset\subset \Omega$, and*
$$\|u\|_{W^{k+2,2}(\Omega')} \leqslant C\, \big(\|u\|_{L^2(\Omega)} + \|f\|_{W^{k,2}(\Omega)}\big). \tag{42}$$
Here the constant depends on $d, h, \operatorname{dist}(\Omega', \Omega)$.

Boundary regularity.

2. Riesz representation theorem.

We consider the Poisson equation with Dirichlet boundary condition:

$$\triangle u = f \qquad \text{in } \Omega; \qquad u = g \quad \text{on } \partial\Omega \tag{43}$$

where g can be extended to a function on the whole Ω. Our purpose is to establish the following result:

Theorem 5. *Let u be a weak solution with $u - g \in W_0^{1,2}(\Omega)$. If $f \in W^{k,2}(\Omega)$, $g \in W^{k+2,2}(\Omega)$, and Ω be of class C^{k+2}, then*

$$u \in W^{k+2,2}(\Omega), \tag{44}$$

and we have the estimate

$$\|u\|_{W^{k+2,2}(\Omega)} \leqslant C\left(\|f\|_{W^{k,2}(\Omega)} + \|g\|_{W^{k+2,2}(\Omega)}\right). \tag{45}$$

The constant C depends on Ω.

Proof. We only give an outline here.

1. First note that since $g \in W^{k+2,2}(\Omega)$, we can replace u by $u - g$ and reduce the problem to

$$\triangle u = f, \qquad u \in W_0^{1,2}(\Omega). \tag{46}$$

2. We first establish $W^{1,2}$ bound:

$$\|u\|_{W^{1,2}} \leqslant C\left(\|g\|_{W^{1,2}} + \|f\|_{L^2}\right). \tag{47}$$

To see this, use $v = u - g$ as the test function. We obtain

$$\left|\int \nabla u \cdot \nabla(u - g)\right| = \left|\int f\,(u - g)\right| \tag{48}$$

therefore

$$\int |\nabla u|^2 \leqslant \left|\int \nabla u \cdot \nabla g\right| + \left|\int f\,(u - g)\right| \leqslant \frac{1}{4}\int |\nabla u|^2 + \int |\nabla g|^2 + \frac{1}{\varepsilon}\int f^2 + \varepsilon\int |u - g|^2. \tag{49}$$

Apply Poincaré's inequality to the last term and choosing ε to be small enough, we obtain the desired estimate.

3. For any $\Omega' \subset\subset \Omega$, we can estimate $\int_{\Omega'} |\partial_{x_i x_j} u|^2 \leqslant C\left(\int u^2 + \int f^2\right) \leqslant C\left(\|g\|_{W^{1,2}} + \|f\|_{L^2}\right)$. Therefore it suffices to establish the desired estimate in a neighborhood of the bondary $\partial\Omega$.

4. We illustrate the basic idea by assuming part of the boundary is in $x_n = 0$. We try to show the $W^{2,2}$ bound for u in a small half-ball $B_R^+ \equiv B_R \cap \{x_n > 0\}$. Note that once this is done, the boundary, which is compact, can be covered by finitely many such balls.

   First note that $\partial_{x_i} u$ is well defined in B_R^+ and belongs to $L^2(B_R^+)$. Now let η be a cut-off function in $C_0^\infty(B_R)$. For all $j \neq n$, $\triangle_j^{\pm h} u$ is well-defined and we can use the test function $\triangle_j^h(\eta^2 \triangle_j^h u)$ as we did when proving the interior regularity, and obtain the desired bound for all $\partial_{x_i x_j} u$ except $\partial_{x_n x_n} u$.

   Now notice that the equation implies

$$\partial_{x_n x_n} u = f - \sum_{i=1}^{n-1} \partial_{x_i x_i} u \tag{50}$$

and therefore this term enjoys the same bound as other double derivatives.

5. For general Ω, we need to first cover $\partial\Omega$ by small balls, and then do a change of variable on each of the balls to "straighten" that part of the boundary. After doing this, however, the equation does not have the simple form

$$\triangle u = f \tag{51}$$

anymore and proving the estimate becomes as difficult as proving similar estimates for the general case. □

Remark 6. It turns out that when the boundary is smooth, one can actually extend the regularity to $\bar{\Omega}$.

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^∞, and let $g \in C^\infty(\partial\Omega)$, $f \in C^\infty(\Omega)$. Then the Dirichlet problem*

$$\triangle u = f \quad \text{in } \Omega; \qquad u = g \quad \text{on } \partial\Omega, \tag{52}$$

possesses a unique solution u which is $C^\infty(\bar{\Omega})$.

The key to the proof is the embedding $W^{k,p}(\Omega) \subset C^m(\bar{\Omega})$ for $0 \leqslant m < k - \frac{d}{p}$.

L^p Regularity.

Theorem 7. *Let $1 < p < \infty$, $f \in L^p(\Omega)$, and let w be the Newton potential of f. Then $w \in W^{2,p}(\Omega)$, $\triangle w = f$ almost everywhere in Ω, and*

$$\|\nabla^2 w\|_{L^p(\Omega)} \leqslant C(n,p)\, \|f\|_{L^p(\Omega)}. \tag{53}$$

Using this theorem, we can obtain the following interior regularity result.

Theorem 8. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\triangle u = f$, $f \in L^p(\Omega)$, $1 < p < \infty$. Then $u \in W^{2,p}(\Omega')$ for any $\Omega' \subset\subset \Omega$, and*

$$\|u\|_{W^{2,p}(\Omega')} \leqslant C\,(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}). \tag{54}$$

Here $C = C(n, p, \Omega', \Omega)$.