Math 527 Lecture 13 Exercises and Problems

Exercises.

Problems.

Problem 1. Prove that $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$ for all $k, p \in [1, \infty), n$.

Problem 2. (Evans) Assume that Ω is bounded and there exists a smooth vector field $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \cdot \boldsymbol{n} \geqslant 1$ along $\partial\Omega$, where $\boldsymbol{n}$ is the unit outer normal vector. Assume $1 \leqslant p < \infty$. Apply the Gauss-Green Theorem to $\int_{\partial\Omega} |u|^p \boldsymbol{\alpha} \cdot \boldsymbol{n} \, \mathrm{d}S$ and prove the trace inequality

$$\int_{\partial\Omega} |u|^p \, \mathrm{d}S \leqslant C \left[\int_\Omega |Du|^p + |u|^p \, \mathrm{d}x \right] \tag{1}$$

for all $u \in C^1(\overline{\Omega})$.

Problem 3. (Evans) Let Ω be bounded, with C^1 boundary. Show that there is no bounded linear operator $T: L^p(\Omega) \mapsto L^p(\partial\Omega)$ such that $Tu = u|_{\partial\Omega}$ for all $u \in C(\overline{\Omega}) \cap L^p(\Omega)$.

Problem 4. (Evans) Integrate by parts to prove the interpolation inequality:

$$\|Du\|_{L^2} \leqslant C \, \|u\|_{L^2}^{1/2} \, \|D^2 u\|_{L^2}^{1/2} \tag{2}$$

for all $u \in C_0^\infty(\Omega)$. Assume Ω is bounded and $\partial\Omega$ smooth, extend this inequality to $u \in H_0^1(\Omega) \cap H^2(\Omega)$. (Hint: Take $v_k \to u$ in H_0^1 while $w_k \to u$ in H^2)

Problem 5. Use scaling to determine the relation of α and p in the embedding relation

$$\|u\|_{C^\alpha} \leqslant C \, \|u\|_{W^{1,p}} \tag{3}$$

when $p > n$.

Problem*.

Problem 6. Prove the following Poincare inequality: Let $\Omega = \{0 < x_n < L\}$. Then for every $u \in H_0^1(\Omega)$,

$$\|u\|_{L^2} \leqslant \frac{L}{\pi} \, \|\nabla u\|_{L^2}. \tag{4}$$

(Hint: First reduce to 1D case; Then use $\int_0^L \left| u' - u \frac{\varphi'}{\varphi} \right|^2 \geqslant 0$.)

Problem 7. Prove the following Poincare inequality: Let $\Omega \subset \mathbb{R}^N$ be such that $\operatorname{meas}(\Omega) < \infty$. Then for every $u \in H_0^1(\Omega)$,

$$\|u\|_{L^2} \leqslant C \, [\operatorname{meas}(\Omega)]^{1/N} \, \|\nabla u\|_{L^2}. \tag{5}$$

(Hint: Take Fourier transform. And estimate separately $\int_{|\xi| < \rho} |\mathcal{F}u|^2$ and $\int_{|\xi| > \rho} |\mathcal{F}u|^2$, choose a particular ρ).