Sobolev Spaces

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Definition and Properties.

- Sobolev spaces $W^{k,p}(\Omega)$:
$$W^{k,p}(\Omega)=\left(\sum_{|\alpha|\leqslant k}\|D^\alpha u\|_{L^p}^p\right)^{1/p}. \tag{1}$$
  $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$
- Approximation properties:
  - $C^\infty(\Omega)\cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$;
  - $C^\infty(\overline{\Omega})\cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$ when $\partial\Omega$ is compact, locally Lipschitz, and Ω lies (locally) on one side of $\partial\Omega$.

Extensions and Traces.

- Extensions.
  If Ω is bounded and $\partial\Omega$ is C^k, then there is an extension operator $E: W^{k,p}(\Omega)\mapsto W^{k,p}(\mathbb{R}^n)$ which is linear, bounded, and $Eu=u$ in Ω;
- Trace.
  If Ω is bounded and $\partial\Omega$ is C^k, then there is a bounded linear operator $T: W^{k,p}(\Omega)\mapsto W^{k-1,p}(\partial\Omega)$ which satisfies $Tu=u|_{\partial\Omega}$ when $u\in C^{k-1}(\overline{\Omega})\cap W^{k,p}(\Omega)$.

Relations between Spaces.

The trick the understand embedding inequalities as
$$\|u\|_{L^q}\leqslant C\,\|u\|_{W^{1,p}} \tag{2}$$
or interpolation inequalities as
$$\|Du\|_{L^p}\leqslant C\,\|u\|_{L^q}^\alpha\,\|D^2u\|_{L^r}^\beta \tag{3}$$
is through scaling. That is, consider a "bump" function u with height h and width l. For such functions $D^k u\sim h/l^k$ and all Sobolev norms can be represented by powers of h and l.

For example, to see what relation q,p must satisfy in the embedding inequality, we compute
$$\|u\|_{L^q}\sim h\,l^{n/q};\qquad \|u\|_{L^p}\sim h\,l^{n/p};\qquad \|Du\|_{L^p}\sim h\,l^{n/p-1}. \tag{4}$$
So the embedding inequality roughly says
$$h\,l^{n/q}\leqslant C\left[h\,l^{n/p}+h\,l^{n/p-1}\right] \tag{5}$$
or equivalently
$$l^{n/q}\leqslant C\left[l^{n/p}+l^{n/p-1}\right]. \tag{6}$$
Letting $l\nearrow\infty$ we conclude that $q\geqslant p$, while letting $l\searrow 0$ we conclude $\frac{n}{q}\geqslant\frac{n}{p}-1$ or equivalently $\frac{1}{q}\geqslant\frac{1}{p}-\frac{1}{n}$. So we expect
$$\|u\|_{L^q}\leqslant C\,\|u\|_{W^{1,p}} \tag{7}$$
to hold when $q\geqslant p$ and $\frac{1}{q}\geqslant\frac{1}{p}-\frac{1}{n}$.

Remark 1. If we further assume Ω to be bounded, then $l\nearrow\infty$ won't happen and we expect the embedding to be true whenever $q\leqslant q_c$ where q_c is given by $\frac{1}{q_c}=\frac{1}{p}-\frac{1}{n}$.

Remark 2. Scaling argument may fail when L^∞ is involved. For example
$$\|u\|_{L^\infty}\leqslant\|u\|_{W^{1,n}} \tag{8}$$

is not true.

Remark 3. When $q < q_c$, we expect the embedding to be compact. That is, bounded sequences in $W^{1,p}$ will have convergent subsequence in L^q.

Remark 4. Such rough understanding should be enough for our future application of Sobolev spaces in this class. For more details see my 527 notes in 2009.