Math 527 Lecture 12 Exercises and Problems

Exercises.

Exercise 1. Show that when $u \in C^1$, its weak derivatives coincide with classical derivatives.

Problems.

Problem 1. Show through examples that the space

$$V = \{u \in C^1, \nabla u \in L^2\}, \qquad \|u\|_V = \|u\|_{C^0} + \|\nabla u\|_{L^2} \tag{1}$$

is not complete.

Problem 2. Let u minimizer

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x. \tag{2}$$

Derive the equation for u. Then define a reasonable weak solution for this equation.

Problem 3. Prove the following using definition:

$$u(x) = |x| \in W^{1,2}(-1,1); \tag{3}$$

Problem 4. Show that $W^{1,2}(-1,1) \subset C(-1,1)$. That is any function $u \in W^{1,2}(-1,1)$ is in fact continuous. Then use this to show that $u(x) = \begin{cases} 1 & 0 \leqslant x < 1 \\ 0 & -1 < x < 0 \end{cases} \notin W^{1,2}(-1,1)$.

Problem 5. Given the trace inequality

$$\|u\|_{L^2(\partial\Omega)} \leqslant C\, \|u\|_{W^{1,2}(\Omega)} \tag{4}$$

for all $u \in C(\bar{\Omega}) \cap W^{1,2}(\Omega)$, show that the boundary value for functions in $W^{1,2}(\Omega)$ are well-defined.

Problem 6. Prove: $W_0^{1,2}(\Omega)$ is a closed subspace of $W^{1,2}(\Omega)$

Problem 7. Write a complete proof of the existence of weak solution for the Laplace equation

$$-\triangle u = 0 \text{ in } \Omega; \qquad u = g \text{ on } \partial\Omega \tag{5}$$

Fill in all the details left out in the lecture note.

Problem*.

Problem.