Weak Solutions

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Dirichlet Principle.

If we multiply the equation by any $v \in C_0^\infty(\Omega)$ and integrate, we have

$$0 = \int (\triangle u)\, v = -\int \nabla u \cdot \nabla v. \tag{1}$$

As a consequence, we have

$$\int |\nabla (u+v)|^2 = \int |\nabla u|^2 + \int |\nabla v|^2 \geqslant \int |\nabla u|^2. \tag{2}$$

In other words, u is the minimizer of the function

$$I(u) \equiv \int_\Omega |\nabla u|^2 \,\mathrm{d}x. \tag{3}$$

Conversely, if u is a minimizer, then for any $v \in C_0^\infty$, and $t > 0$, we have

$$\int |\nabla (u + t\, v)|^2 \geqslant \int |\nabla u|^2 \Longleftrightarrow t^2 \int |\nabla v|^2 - 2\,t \int \nabla u \cdot \nabla v \geqslant 0 \tag{4}$$

which implies

$$\int (\triangle u)\, v = -\int \nabla u \cdot \nabla v = 0 \tag{5}$$

by taking $t \searrow 0$ and consequently

$$\triangle u = 0 \tag{6}$$

when $u \in C^2$.

Weak solution: first try.

Notice: For $I(u)$ to be well-defined, we only need the existence of ∇u, and we do not need $u \in C^2$. If we drop the requirement of $u \in C^2$, the above argument gives:

$$u = \operatorname{argmin} I(u) \Longleftrightarrow \int \nabla u \cdot \nabla v = 0 \tag{7}$$

for all v with ∇v exists and $v = 0$ along $\partial \Omega$.

Thus we can define "weak solution" using $\int \nabla u \cdot \nabla v = 0$ instead of $-\triangle u = 0$. The hope is that the existence of thus defined weak solution would be easy to establish, through showing the existence of minimizer for $I(u)$.

Existence of minimizer for $I(u)$.

We use the so-called "direct method", which consists of 3 steps:

1. Take a minimizing sequence u_n;
2. Establish that u_n is Cauchy in certain space;
3. Show that the limit u is in fact a minimizer.

Let u_n be a minimizing sequence, that is

$$\lim_{n \to \infty} I(u_n) = \inf I(u). \tag{8}$$

Then one calculates

$$\begin{aligned} \int |\nabla u_n - \nabla u_m|^2 &= \int |\nabla u_n|^2 - 2\,\nabla u_n \cdot \nabla u_m + |\nabla u_m|^2 \\ &= 2\int |\nabla u_n|^2 + 2\int |\nabla u_m|^2 - \int |\nabla u_n + \nabla u_m|^2 \\ &= 2\,I(u_n) + 2\,I(u_m) - 4\,I\Big(\frac{u_n + u_m}{2}\Big). \end{aligned} \tag{9}$$

Note that we divide $u_n + u_m$ by 2 to make sure the boundary condition is satisfied.

Since

$$4\, I\Big(\frac{u_n + u_m}{2}\Big) \geqslant 4 \inf I(u) = \lim\, [2\, I(u_n) + 2\, I(u_m)], \tag{10}$$

we see that

$$\int |\nabla u_n - \nabla u_m|^2 \to 0 \qquad n, m \to \infty \tag{11}$$

or equivalently $\{\nabla u_n\}$ is a Cauchy sequence in the space L^2 of all square integrable functions. Thus there is a limit function $w = \lim \nabla u_n$ which is square integrable.

From the above we conclude:

- The weak solution u should satisfy $\nabla u \in L^2$;

$W^{1,2}(\Omega)$.

Based on the above, we try the following space:

$$V = \{u \in C^1, \nabla u \in L^2\}, \qquad \|u\|_V = \|u\|_{C^0} + \|\nabla u\|_{L^2} \tag{12}$$

However this space is not complete, and we have to switch to

$$W^{1,2}(\Omega) := \{u \in L^2, \nabla u \in L^2\}, \qquad \|u\|_{W^{1,2}} := \|u\|_{L^2} + \|\nabla u\|_{L^2} \tag{13}$$

or equivalently but often more convenient to use:

$$\|u\|_{W^{1,2}} := (\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)^{1/2}. \tag{14}$$

However this leads to the following problem: If u is not a priori C^1, how do we talk about ∇u? The answer is weak derivatives.

Weak derivatives.

The weak derivative for a function u is a function v such that

$$\int_\Omega u\, \partial_j \phi = -\int_\Omega v\, \phi \tag{15}$$

for all $\phi \in C_0^\infty$. We denote v by $\partial_j u$.

Boundary value.

One important property of $W^{1,2}(\Omega)$ is that when $\partial\Omega$ is nice (say C^1), then $C(\bar{\Omega}) \cap W^{1,2}(\Omega)$ is dense in it. Combining with the trace inequality:

$$\|u\|_{L^2(\partial\Omega)} \leqslant C\, \|u\|_{W^{1,2}(\Omega)} \tag{16}$$

for all $u \in C(\bar{\Omega}) \cap W^{1,2}(\Omega)$, we can define boundary values for all $u \in W^{1,2}(\Omega)$.

$W_0^{1,2}(\Omega)$.

An important special subspace of $W^{1,2}(\Omega)$ is

$$W_0^{1,2}(\Omega) := \{u \in W^{1,2}(\Omega), u = 0 \text{ on } \partial\Omega\} \tag{17}$$

whose norm is the same as that of $W^{1,2}$.

- $W_0^{1,2}(\Omega)$ is a closed subspace of $W^{1,2}(\Omega)$ and therefore is a Banach space itself.
- $W_0^{1,2} = \overline{C_0^\infty(\Omega)}$ where the closure is under the $W^{1,2}$ norm.

Weak solution.

The real definition for weak solution for

$$-\triangle u = 0 \text{ in } \Omega; \qquad u = g \text{ on } \partial\Omega. \tag{18}$$

Definition 1. *$u \in W^{1,2}(\Omega)$ is a weak solution if*

- *$u = g$ on $\partial\Omega$;*

- *For any $v \in W_0^{1,2}(\Omega)$, we have*

$$\int \nabla u \cdot \nabla v = 0. \tag{19}$$

Existence of weak solution.

We have shown the following: If u_n is a minimizing sequence of $I(u) := \int |\nabla u|^2$, then there is $w \in L^2$ such that $\nabla u_n \longrightarrow w$ in L^2, and $\int w^2 \leqslant \inf I(u)$.

So all we need to do is to show that $w = \nabla u$ for some $u \in W^{1,2}(\Omega)$ with $u = g$ on $\partial\Omega$. It suffices to show $u_n \to u$ in $W^{1,2}(\Omega)$.

To show this we need the following Poincare inequality:

Theorem 2. *There is a constant C, depending on the bounded set Ω only, such that for all $u \in W_0^{1,2}(\Omega)$, we have*

$$\|u\|_{L^2(\Omega)} \leqslant C \, \|\nabla u\|_{L^2(\Omega)}. \tag{20}$$

Proof. Extend u by 0 outside Ω we obtain a $W^{1,2}$ function, still denoted u, defined on the box. Without loss of generality we assume the box is $0 \leqslant x_i \leqslant R$. Integrating from $x_n = 0$ we have

$$u(x_1, \ldots, x_n) = \int_0^{x_n} \partial_{x_n} u(x_1, \ldots, x_{n-1}, t) \, \mathrm{d}t. \tag{21}$$

Now we have

$$\begin{aligned} \int |u|^2 &\leqslant \int \left(|u(x_1, \ldots, x_n)| \int_0^{x_n} |\nabla u| \, \mathrm{d}t \right) \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &\leqslant \int \left(|u(x_1, \ldots, x_n)| \int_0^{R} |\nabla u| \, \mathrm{d}t \right) \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \int \int_0^R |u(x_1, \ldots, x_n)| \, |\nabla u(x_1, \ldots, t)| \, \mathrm{d}t \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &\leqslant \left(\int \int_0^R |u(x_1, \ldots, x_n)|^2 \right)^{1/2} \left(\int \int |\nabla u(x_1, \ldots, t)| \, \mathrm{d}t \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \right)^{1/2} \\ &\leqslant R^{1/2} \left(\int |u|^2 \right)^{1/2} R^{1/2} \left(\int |\nabla u|^2 \right)^{1/2}. \end{aligned} \tag{22}$$

and thus obtaining

$$\|u\|_{L^2(\Omega)} \leqslant R \, \|\nabla u\|_{L^2(\Omega)}. \tag{23}$$

□

Remark 3. A more refined (and more general, as it can be applied to unbounded regions) estimate has the constant

$$C = \left(\frac{|\Omega|}{\alpha(n)} \right)^{1/n} \tag{24}$$

where $|\Omega|$ is the volume of Ω and $\alpha(n)$ is the volume of the n-dimensional unit ball. See the proof of Theorem 7.2.2 in J. Jost **Partial Differential Equations**.