Math 527 Lecture 11 Exercises and Problems

Exercises.

Exercise 1. Prove the following:

If $f_1, f_2 \in C^\alpha(\Omega)$, then $f_1 f_2 \in C^\alpha(G)$, and

$$|f_1 f_2|_{C^\alpha} \leqslant \left(\sup_\Omega |f_1|\right) |f_2|_{C^\alpha} + \left(\sup_\Omega |f_2|\right) |f_1|_{C^\alpha}. \tag{1}$$

Hint: Use the identity $f_1(x) f_2(x) - f_1(y) f_2(y) = f_1(x) [f_2(x) - f_2(y)] + f_2(y) [f_1(x) - f_1(y)]$.

Problems.

Problem 1. Write down the detailed proof of

If $f \in L^\infty(\Omega)$, then $u \in C^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$, and

$$\|u\|_{C^{1,\alpha}(\Omega)} \leqslant c \, \|f\|_{L^\infty(\Omega)}. \tag{2}$$

where

$$u(x) \equiv \int_\Omega \Phi(x - y) \, f(y) \, \mathrm{d}y. \tag{3}$$

Problem*.

Problem. Give detailed proof of

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and $\Omega_0 \Subset \Omega$. Let u solve $\triangle u = f$ in Ω.*

a) If $f \in C^0(\Omega)$, then $u \in C^{1,\alpha}(\Omega)$ for any $\alpha \in (0,1)$, and

$$\|u\|_{C^{1,\alpha}(\Omega_0)} \leqslant c \, (\|f\|_{C^0(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{4}$$

b) If $f \in C^\alpha(\Omega)$ for $0 < \alpha < 1$, then $u \in C^{2,\alpha}(\Omega)$, and

$$\|u\|_{C^{2,\alpha}(\Omega_0)} \leqslant c \, (\|f\|_{C^\alpha(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{5}$$

Here

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega u^2\right)^{1/2}. \tag{6}$$