Regularity for Poisson Equation

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Intuitively, the solution u to the Poisson equation

$$\triangle u = f \tag{1}$$

should have better regularity than the right hand side f. In particular one expects u to be "twice more differentiable" than f. The validity of this conjecture depends on the function spaces we are looking at.

Note. "Schauder Theory" in fact denotes the similar results for the general linear elliptic PDE

$$\sum a_{ij}(x) \frac{\partial u}{\partial x_i \partial x_j} + \sum b_i(x) \frac{\partial u}{\partial x_i} + c(x)\, u(x) = 0. \tag{2}$$

Nevertheless we use it (instead of "$C^{2,\alpha}$ estimates") as the title of this lecture to make it easy to display on the web.

1. Counter-examples.

The most "natural" conjecture one would make is $f \in C(\Omega) \Longrightarrow u \in C^2(\Omega)$. Anyway, it is indeed true in 1D. However it cease to be true when the dimension is bigger than 1.

Example 1. ($f \in L^\infty$ but $u \notin C^{1,1}$).

$$u(x_1, x_2) = |x_1|\, |x_2| \log(|x_1| + |x_2|). \tag{3}$$

We compute (in $x_1, x_2 > 0$)

$$\frac{\partial^2 u}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left[x_2 \log(x_1 + x_2) + \frac{x_1 x_2}{x_1 + x_2} \right] = \frac{2\, x_2}{x_1 + x_2} - \frac{x_1 x_2}{(x_1 + x_2)^2}; \tag{4}$$

$$\frac{\partial^2 u}{\partial x_2^2} = \frac{2\, x_1}{x_1 + x_2} - \frac{x_1 x_2}{(x_1 + x_2)^2}. \tag{5}$$

Thus

$$\triangle u = 2 - \frac{2\, x_1 x_2}{(x_1 + x_2)^2} \tag{6}$$

and the RHS is a bounded function.

However, we compute

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = \log(x_1 + x_2) + 1 - \frac{x_1 x_2}{(x_1 + x_2)^2} \notin L^\infty. \tag{7}$$

Example 2. (f continuous but $u \notin C^{1,1}$).

$$\triangle u = f(x) \equiv \frac{x_2^2 - x_1^2}{2\, |x|^2} \left[\frac{n+2}{(-\log |x|)^{1/2}} + \frac{1}{2\, (-\log |x|)^{3/2}} \right], \qquad x \in B_R \subset \mathbb{R}^n. \tag{8}$$

$f(x)$ is continuous after setting $f(0) = 0$.

However, the solution

$$u(x) = (x_1^2 - x_2^2)\, (-\log |x|)^{1/2} \tag{9}$$

has

$$\frac{\partial^2 u}{\partial x_1^2} \longrightarrow \infty \qquad x \to 0. \tag{10}$$

Therefore $u \notin C^{1,1}$.[1]

1. One can show that there is no classical solution to this problem. Assume otherwise a classical solution v exists, then the difference $u - v$ is a bounded harmonic function in $B_R \backslash \{0\}$. One can show that such functions can be extended as a harmonic function in the whole B_R which means $\nabla^2 u$ must be bounded, a contradiction.

2. C^α regularity.

The right space to work on are the Hölder spaces.

Definition 3. (Hölder continuity) *Let $f\colon \Omega \mapsto \mathbb{R}$, $x_0 \in \Omega$, $0 < \alpha < 1$. The function f is called Hölder continuous at x_0 with exponent α if*

$$\sup_{x\in\Omega} \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha} < \infty. \tag{11}$$

f is called Hölder continuous in Ω if it is Hölder continuous at each $x_0 \in \Omega$ (with the same exponent α), denoted $f \in C^\alpha(\Omega)$.

When $\alpha = 1$, f is called Lipschitz continuous at x_0, denoted $f \in \mathrm{Lip}(\Omega)$ or $f \in C^{0,1}(\Omega)$.

$C^{k,\alpha}(\bar{\Omega})$ contains $f \in C^k(\bar{\Omega})$ whose kth derivatives are uniformly Hölder continuous with exponent α over $\bar{\Omega}$, that is

$$\sup_{x,y\in\bar{\Omega}} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty. \tag{12}$$

$C^{k,\alpha}(\Omega)$ contains $f \in C^k(\Omega)$ whose kth derivatives are uniformly Hölder continuous with exponent α in every compact subset of Ω.

Example 4. The functions $f(x) = |x|^\alpha$, $0 < \alpha < 1$, is Hölder continuous with exponent α at $x = 0$. It is Lipschitz continuous when $\alpha = 1$.

Remark 5. When $k = 0$, we usually use C^α for $C^{0,\alpha}$ since there is no ambiguity for $0 < \alpha < 1$.

We can define the seminorm

$$|f|_{C^\alpha(\bar{\Omega})} \equiv \sup_{x,y\in\Omega} \frac{|f(x) - f(y)|}{|x - y|^\alpha}, \tag{13}$$

and the norms

$$\|f\|_{C^\alpha(\bar{\Omega})} = \|f\|_{C^0(\Omega)} + |f|_{C^\alpha(\bar{\Omega})}, \tag{14}$$

$$\|f\|_{C^{k,\alpha}(\bar{\Omega})} = \sum_{|\alpha|\leqslant k} \|\partial^\alpha f\|_{C^0(\Omega)} + \sum_{|\alpha|=k} |\partial^\alpha f|_{C^\alpha(\bar{\Omega})}. \tag{15}$$

where

$$\|f\|_{C^0(\bar{\Omega})} = \sup_{x\in\Omega} |f|. \tag{16}$$

The following property is important. In short, C^α is an algebra.

Lemma 6. *If $f_1, f_2 \in C^\alpha(\Omega)$, then $f_1 f_2 \in C^\alpha(G)$, and*

$$|f_1 f_2|_{C^\alpha} \leqslant \left(\sup_\Omega |f_1|\right) |f_2|_{C^\alpha} + \left(\sup_\Omega |f_2|\right) |f_1|_{C^\alpha}. \tag{17}$$

Proof. Left as exercise. □

Theorem 7. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded,*

$$u(x) \equiv \int_\Omega \Phi(x - y)\, f(y)\, \mathrm{d}y, \tag{18}$$

where Φ is the fundamental solution. Then

a) If $f \in C_0^\alpha(\bar{\Omega})$, $0 < \alpha < 1$, then $u \in C^{2,\alpha}(\bar{\Omega})$, and

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leqslant c\, \|f\|_{C^\alpha(\bar{\Omega})}. \tag{19}$$

b) If $f \in L^\infty(\Omega)$ ($\alpha = 0$ case), then $u \in C^{1,\alpha}(\bar{\Omega})$ for any $0 < \alpha < 1$, and

$$\|u\|_{C^{1,\alpha}(\bar{\Omega})} \leqslant c\, \|f\|_{L^\infty(\bar{\Omega})}. \tag{20}$$

c) *If* $f \in \mathrm{Lip}(\bar{\Omega})$ *($\alpha = 1$ case) with support contained in* $\bar{\Omega}$*, then* $u \in C^{2,\alpha}(\bar{\Omega})$ *for any* $0 < \alpha < 1$*, and*

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leqslant c\,\|f\|_{\mathrm{Lip}(\bar{\Omega})}. \tag{21}$$

Proof.

a) Recall that $\Phi(x-y) = C\log|x-y|$ for $n=2$ and $\Phi(x-y) = C\,\frac{1}{|x-y|^{n-2}}$. for $n \geqslant 3$.

1. We first show $u \in C^1$.

   Formally differentiating we obtain

$$\partial_{x_i} u = \int_\Omega (\partial_{x_i}\Gamma(x,y))\, f(y)\,\mathrm{d}y = C\int_\Omega \frac{x_i - y_i}{|x-y|^n}\, f(y)\,\mathrm{d}y. \tag{22}$$

   It is easy to check that the integrand is integrable. Therefore by the theorem regarding differentiating with respect to a parameter for Lebesgue integrals, we see that the formal relation

$$\partial_{x_i} u = C\int_\Omega \frac{x_i - y_i}{|x-y|^n}\, f(y)\,\mathrm{d}y \tag{23}$$

   indeed holds.

2. Next we show $u \in C^{2,\alpha}$. In the following we will omit the constant factor C. In this step we do some preparations.

   Again formally differentiating, we obtain

$$\partial_{x_i x_j} u = \int \left(\frac{\delta_{ij}}{|x-y|^n} - \frac{n\,(x_i - y_i)\,(x_j - y_j)}{|x-y|^{n+2}}\right) f(y)\,\mathrm{d}y. \tag{24}$$

   But this time the integrand is not automatically integrable and therefore this equality is dubious. To overcome this difficulty, we first work in the weak sense.

   By extending f outside Ω to be 0 (resulting in a distribution with compact support), we can write

$$\partial_{x_i} u = \frac{x_i}{|x|^n} * f \tag{25}$$

   in the sense of distributions. Thus we have

$$\partial_{x_i x_j} u = \left[\partial_{x_j}\left(\frac{x_i}{|x|^n}\right)\right] * f \tag{26}$$

   in the sense of distributions. We compute the distributional derivative $\partial_{x_j}\left(\frac{x_i}{|x|^n}\right)$ now.

   Take any $\phi \in C_0^\infty(\mathbb{R}^n)$, we know

$$\left[\partial_{x_j}\left(\frac{x_i}{|x|^n}\right)\right](\phi) = -\int \frac{x_i}{|x|^n}\,(\partial_{x_j}\phi)(x)\,\mathrm{d}x = -\lim_{\varepsilon \searrow 0}\int_{\mathbb{R}^n \backslash B_\varepsilon} \frac{x_i}{|x|^n}\,(\partial_{x_j}\phi)\,\mathrm{d}x. \tag{27}$$

   Now integrate by parts, we have

$$\begin{aligned} -\int_{\mathbb{R}^n \backslash B_\varepsilon} \frac{x_i}{|x|^n}\,(\partial_{x_j}\phi)\,\mathrm{d}x &= -\int_{\partial B_\varepsilon} \phi(x)\,\frac{x_i}{|x|^n}\left(-\frac{x_j}{|x|}\right) + \int_{\mathbb{R}^n \backslash B_\varepsilon} S_{ij}(x)\,\phi(x)\,\mathrm{d}x \\ &= \int_{|x|=\varepsilon} \phi(x)\,\frac{x_i\,x_j}{\varepsilon^{n+1}} + \int_{\mathbb{R}^n \backslash B_\varepsilon} S_{ij}(x)\,\phi(x)\,\mathrm{d}x. \end{aligned} \tag{28}$$

   where

$$S_{ij}(x) = \frac{\delta_{ij}}{|x-y|^n} - \frac{n\,(x_i - y_i)\,(x_j - y_j)}{|x-y|^{n+2}} \tag{29}$$

   is the formal derivative. For the boundary term, we write

$$\int_{|x|=\varepsilon} \phi(x)\,\frac{x_i\,x_j}{\varepsilon^{n+1}} = \phi(0)\int_{|x|=\varepsilon} \frac{x_i\,x_j}{\varepsilon^{n+1}} + \int_{|x|=\varepsilon} [\phi(x) - \phi(0)]\,\frac{x_i\,x_j}{\varepsilon^{n+1}}. \tag{30}$$

Note that since $\phi \in C_0^{\infty}$, $\phi(x) - \phi(0) = O(|x|) = O(\varepsilon)$ which makes the second term an $O(\varepsilon)$ quantity. For the first term, a symmetry argument shows that the integral vanishes when $i \neq j$. When $i = j$, we use symmetry and the fact that

$$\int_{|x|=\varepsilon} \sum \frac{x_i\, x_i}{|x|^{n+1}} = \int_{|x|=\varepsilon} \frac{1}{\varepsilon^{n-1}} = \omega_{n-1}, \tag{31}$$

where ω_{n-1} is the surface area of the $n-1$ dimensional unit sphere, to conclude that the limit is $c\,\phi(0)$ for some constant c.

Therefore we have shown that

$$\left[\partial_{x_j}\left(\frac{x_i}{|x|^n}\right)\right](\phi) = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n \backslash B_\varepsilon} S_{ij}(x)\, \phi(x)\, \mathrm{d}x + c\,\delta \tag{32}$$

As a consequence, we have

$$\partial_{x_i x_j} u(x) = \lim_{\varepsilon \searrow 0} \int_{\Omega \backslash B_\varepsilon} S_{ij}(x-y)\, f(y)\, \mathrm{d}y + c\, f(x). \tag{33}$$

We now show directly that the second derivative $\partial_{x_i x_j} u$ is Hölder continuous with power α. Since $f(x) \in C^{\alpha}$, we only need to show that

$$\lim_{\varepsilon \searrow 0} \left[\int_{\Omega \backslash B_\varepsilon(x_1)} S_{ij}(x_1 - y)\, f(y)\, \mathrm{d}y - \int_{\Omega \backslash B_\varepsilon(x_2)} S_{ij}(x_2 - y)\, f(y)\, \mathrm{d}y\right] |x_1 - x_2|^{-\alpha} < \infty. \tag{34}$$

uniformly for $x_1, x_2 \in \Omega$.

3. $\partial_{x_i x_j} u \in C^{\alpha}$.

Inspection of S_{ij} reveals that for any $0 < R_1 < R_2$:

$$\int_{R_1 \leqslant |y| \leqslant R_2} S_{ij}(x-y)\, \mathrm{d}y = 0. \tag{35}$$

To make things simple, we extend f to be 0 outside Ω. The resulting function is in $C_0^{\alpha}(\mathbb{R}^n)$ [2]. We have

$$\int_{\mathbb{R}^n \backslash B^\varepsilon} S_{ij}(x-y) f(y)\, \mathrm{d}y = \int_{\mathbb{R}^n \backslash B^\varepsilon} S_{ij}(x-y)\, [f(y) - f(x)]\, \mathrm{d}y. \tag{36}$$

Note that since $f \in C^{\alpha}$, the integrand is integrable now, which means $u \in C^2$ has been proved.

To show $u \in C^{2,\alpha}$ we need more refined analysis of the integral. First note that in writing the quantity as

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n \backslash B^\varepsilon} S_{ij}(x-y)\, [f(y) - f(x)]\, \mathrm{d}y \tag{37}$$

the singularity has been removed and we can take the limit and write

$$\partial_{x_i x_j} u(x) = \int_{\mathbb{R}^n} S_{ij}(x-y)\, [f(y) - f(x)]\, \mathrm{d}y. \tag{38}$$

For any $x_1, x_2 \in \Omega$, setting $\delta = 2\,|x_1 - x_2|$, we have

$$\begin{aligned} \partial_{x_i x_j} u(x_1) - \partial_{x_i x_j} u(x_2) &= \int_{\mathbb{R}^n} S_{ij}(x_1 - y)\, [f(y) - f(x_1)] - S_{ij}(x_2 - y)\, [f(y) - f(x_2)] \\ &= \int_{B_\delta(x_1)} + \int_{\mathbb{R}^n \backslash B_\delta(x_1)} \equiv A + B. \end{aligned} \tag{39}$$

For A, we bound $|f(y) - f(x_1)| \leqslant \|f\|_{C^\alpha}\, |y - x_1|^{\alpha}$ and $|f(y) - f(x_2)| \leqslant \|f\|_{C^\alpha}\, |y - x_2|^{\alpha}$, and get

$$|A| \leqslant C\, \|f\|_{C^\alpha}\, \delta^{\alpha} = C\, \|f\|_{C^\alpha}\, |x_1 - x_2|^{\alpha}. \tag{40}$$

2. Let $\tilde{f}$ be the extended function. Then one notices that $|\tilde{f}(x) - \tilde{f}(y)| = |f(x) - f(y)|$ when $x, y \in \Omega$, vanishes when x, $y \neq \Omega$, and equals $|f(x) - f(y')|$ when $x \in \Omega$ and $y \notin \Omega$, where y' is the intersection of $\partial\Omega$ and the line connecting x, y.

For B, we have

$$\begin{aligned} B &= \int_{\mathbb{R}^n\setminus B_\delta(x_1)} S_{ij}(x_1-y)\,[f(y)-f(x_1)] - S_{ij}(x_2-y)\,[f(y)-f(x_2)] \\ &= \int_{\mathbb{R}^n\setminus B_\delta(x_1)} S_{ij}(x_1-y)\,[f(x_2)-f(x_1)]\,\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^n\setminus B_\delta(x_1)} [S_{ij}(x_1-y)-S_{ij}(x_2-y)]\,[f(y)-f(x_2)]\,\mathrm{d}y \\ &\equiv B_1+B_2. \end{aligned} \tag{41}$$

It is easy to see that $B_1=0$. For B_2, we estimate[3]

$$|S_{ij}(x_1-y)-S_{ij}(x_2-y)| \leqslant |\nabla S_{ij}(x_3-y)|\,|x_1-x_2| \leqslant C\,\frac{|x_1-x_2|}{|x_3-y|^{n+1}} \tag{42}$$

for some x_3 lying on the line segment connecting x_1, x_2. We have

$$\begin{aligned} |B_2| &\leqslant C\,\|f\|_{C^\alpha} \int_{\mathbb{R}^n\setminus B_\delta(x_1)} \frac{|x_1-x_2|}{|x_3-y|^{n+1}}\,|y-x_2|^\alpha \\ &\leqslant C\,\|f\|_{C^\alpha}\,|x_1-x_2| \int_{\mathbb{R}^n\setminus B_\delta(x_1)} |x_1-y|^{\alpha-(n+1)}\,\mathrm{d}y \\ &= C\,\|f\|_{C^\alpha}\,|x_1-x_2|\,|x_1-x_2|^{\alpha-1} \\ &= C\,\|f\|_{C^\alpha}\,|x_1-x_2|^\alpha. \end{aligned} \tag{43}$$

where we have used the fact that $|x_i-y|$ are all comparable $(i=1,2,3)$ for $y\notin B_\delta(x_1)$.

b) We prove the stronger statement $\partial_{x_i}u$ is Log-Lipschitz, that is

$$|\partial_{x_i}u(x_1)-\partial_{x_i}u(x_2)| \leqslant C\sup|f|\,|x_1-x_2|\log(|x_1-x_2|^{-1}). \tag{44}$$

It is easy to get

$$|\partial_{x_i}u(x_1)-\partial_{x_i}u(x_2)| \leqslant \sup_\Omega |f| \int_\Omega \left|\frac{(x_1-y)_i}{|x_1-y|^n}-\frac{(x_2-y)_i}{|x_2-y|^n}\right| \mathrm{d}y. \tag{45}$$

We extend f by 0 and break the integral to $\int_{B_\delta(x_1)}+\int_{\mathbb{R}^n\setminus B_\delta(x_1)}$ with $\delta=2\,|x_1-x_2|$. For the first term we obtain a bound $C\sup_\Omega|f|\,|x_1-x_2|$, for the second we use

$$\left|\frac{(x_1-y)_i}{|x_1-y|^n}-\frac{(x_2-y)_i}{|x_2-y|^n}\right| \leqslant C\,\frac{|x_1-x_2|}{|x_3-y|^n} \tag{46}$$

with a uniform C. Now note that for R big enough, $\int_{\mathbb{R}^n\setminus B_\delta}=\int_{B_R\setminus B_\delta(x_1)}\leqslant\int_{B_R\setminus B_{\delta/2}(x_3)}$. The integration can be carried out explicitly and yields the bound

$$C\,|x_1-x_2|\,(\log R-\log|x_1-x_2|). \tag{47}$$

Thus ends the proof (the details are left as exercise).

c) This part is the same as b). Omitted. □

Remark 8. The techniques involved in the above proof is standard in the theory of singular integrals and are applied extensively in equations arising from fluid mechanics, mathematical biology, etc.

Remark 9. One may notice that when $f\in L^\infty$, one cannot reach $\partial_{x_i}u\in \mathrm{Lip}$ (that is $\partial_{x_ix_j}u\in L^\infty$). The reason is that the operator $\partial_{x_ix_j}(-\triangle)^{-1}$ does not map L^∞ into L^∞. Details can be found in any textbook in real harmonic analysis.

3. Note that the intermediate value theorem gives x_3 depending on y. But when we are working outside $B_\delta(x_1)$, $|\xi-y|$ are all comparable for any ξ between x_1 and x_2.

When f does not have compact support, we cannot obtain uniform bounds for u over the whole Ω, but we can obtain estimates on any smaller set $\Omega_0 \subset\subset \Omega$. [4]

Theorem 10. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and $\Omega_0 \Subset \Omega$. Let u solve $\triangle u = f$ in Ω.*

a) If $f \in C^0(\Omega)$, then $u \in C^{1,\alpha}(\Omega)$ for any $\alpha \in (0,1)$, and

$$\|u\|_{C^{1,\alpha}(\Omega_0)} \leqslant c\,(\|f\|_{C^0(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{48}$$

b) If $f \in C^\alpha(\Omega)$ for $0 < \alpha < 1$, then $u \in C^{2,\alpha}(\Omega)$, and

$$\|u\|_{C^{2,\alpha}(\Omega_0)} \leqslant c\,(\|f\|_{C^\alpha(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{49}$$

Here

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega u^2\right)^{1/2}. \tag{50}$$

Proof. We just give an outline of the proof here. Set η be a cut-off function and consider $\phi = \eta\,u$. We have

$$\triangle\phi = F \equiv \eta\,f + 2\,\nabla u \cdot \nabla\eta + u\,\triangle\eta \tag{51}$$

where the RHS has compact support.

This gives

$$\|F\|_{L^\infty} \leqslant c(\eta)\,\|f\|_{L^\infty} + C(\eta)\,\|u\|_{C^1}] \tag{52}$$

and

$$\|F\|_{C^\alpha} \leqslant c(\eta)\,\|f\|_{C^\alpha} + C(\eta)\,\|u\|_{C^{1,\alpha}}]. \tag{53}$$

Next we show that for any $\varepsilon > 0$, there is $N(\varepsilon) > 0$ such that

$$\|u\|_{C^1} \leqslant N(\varepsilon)\,\|u\|_{L^2} + \varepsilon\,\|u\|_{C^{1,\alpha}} \tag{54}$$

and

$$\|u\|_{C^{1,\alpha}} \leqslant N(\varepsilon)\,\|u\|_{L^2} + \varepsilon\,\|u\|_{C^{2,\alpha}}. \tag{55}$$

This is shown via reductio ad absurdum using the Arzela-Ascoli theorem.

Thus we obtain

$$\|u\|_{C^{1,\alpha}(\Omega_0)} \leqslant C(\eta)\,[\varepsilon\,\|u\|_{C^{1,\alpha}(\Omega)} + \|u\|_{L^2(\Omega)}] + c(\eta)\,N(\varepsilon)\,\|f\|_{C^0(\Omega)} \tag{56}$$

(and a similar estimate for $\|u\|_{C^{2,\alpha}(\Omega_0)}$) with the problem that the $C^{1,\alpha}$ norm on the LHS is on Ω_0 while that on the RHS is on a bigger set Ω and therefore cannot be absorbed into the LHS.

This difficulty is overcome by the following technical trick. Consider the case when $\Omega_0 = B_r$, $\Omega = B_{R_2}$, we have

$$\|u\|_{C^{1,\alpha}(B_r)} \leqslant C(\eta)\left[\varepsilon\,\|u\|_{C^{1,\alpha}(B_{R_2})} + \|u\|_{L^2(B_{R_2})}\right] + c(\eta)\,N(\varepsilon)\,\|f\|_{C^0(B_{R_2})}. \tag{57}$$

Now set[5]

$$A \equiv \sup_{0 \leqslant r \leqslant R} (R - r)^3\,\|u\|_{C^{1,\alpha}(B_r)}. \tag{58}$$

for some $R > R_2$.

Now choose R_1 such that

$$A_1 \leqslant 2\,(R - R_1)^3\,\|u\|_{C^{1,\alpha}(B_{R_1})}, \tag{59}$$

This gives

$$\begin{aligned} A_1 &\leqslant 2\,(R - R_1)^3\,\|u\|_{C^{1,\alpha}(B_{R_1})} \\ &\leqslant 2\,(R - R_1)^3\left[\varepsilon\,C(\eta)\,\|u\|_{C^{1,\alpha}(B_{R_2})} + C(\eta)\,\|u\|_{L^2(B_{R_2})}\right] \\ &\quad + 2\,(R - R_1)^3\,c(\eta)\,N(\varepsilon)\,\|f\|_{C^0(B_{R_2})}. \end{aligned} \tag{60}$$

4. Meaning: The closure $\overline{\Omega_0}$ is a compact subset of Ω.

5. Here it seems we need to assume the finiteness of this quantity.

Now observe that $C(\eta)\sim\frac{1}{(R_2-R_1)^2}$ and $c(\eta)\sim 1$, we have, using the definition of A_1,

$$A_1\leqslant C\,\frac{(R-R_1)^3}{(R-R_2)^3}\,\frac{\varepsilon}{(R_1-R_2)^2}\,A_1+C'\,N(\varepsilon)\,\frac{(R-R_1)^3}{(R_2-R_1)^2}\,\|u\|_{L^2(B_{R_2})}+C''\,(R-R_1)^3\,\|f\|_{C^0(B_{R_2})}. \tag{61}$$

Now for fixed R,R_1, one can choose R_2 and ε appropriately so that the coefficient of A_1 on the RHS is less than 1. Thus we obtain the desired estimate for

$$\|u\|_{C^{1,\alpha}(B_r)}\leqslant\frac{1}{(R-r)^3}\,A_1. \tag{62}$$

Now we can cover Ω_0 by balls B_r, and set $R=r+d$ where $d=\operatorname{dist}(\Omega_0,\partial\Omega)$, and finish the proof. □

Corollary 11. *If u solves $\triangle u=f$ with $f\in C^{k,\alpha}(\Omega)$ for $k\in\mathbb{N}$ and $0<\alpha<1$, then $u\in C^{k+2,\alpha}(\Omega_0)$ for any $\Omega_0\subset\subset\Omega$ and*

$$\|u\|_{C^{k+2,\alpha}(\Omega_0)}\leqslant c\left(\|f\|_{C^{k,\alpha}(\Omega)}+\|u\|_{L^2(\Omega)}\right). \tag{63}$$

In particular, $u\in C^\infty$ when $f\in C^\infty$.

3. Regularity and existence: method of continuity.

We briefly discuss why the regularity estimates matter. Consider two bounded linear operators L, L' from Banach spaces X to Y.[6] Assume that we know that L is surjective and wish to establish that L' is also surjective, in other words the solvability of

$$L'x=y. \tag{64}$$

for arbitrary $y\in Y$.

Define a family of operators

$$L_t=(1-t)\,L+t\,L'. \tag{65}$$

Thus $L_0=L$ and $L_1=L'$.

Assumption. We have uniform (that is, independent of t) a priori (that is, assuming the existence of solutions) estimates

$$\|u\|_X\leqslant c\,\|L_tu\|_Y. \tag{66}$$

Under this assumption, one has

Theorem 12. *If L_0 is surjective, so is L_1.*

Proof. The idea is to show that there is ε independent of t, such that if L_τ is surjective, so is L_t for all $t\in(\tau,\tau+\varepsilon)$.

To see this, note that the estimate $\|u\|_X\leqslant c\,\|L_tu\|_Y$ implies that all L_t's are injective. Thus the inverse L_τ^{-1} is well-defined and bounded.

We write

$$L_tu=f \tag{67}$$

into

$$L_\tau u=f+(L_\tau-L_t)u=f+(t-\tau)\,(L_0-L_1)u. \tag{68}$$

This gives

$$u=L_\tau^{-1}f+(t-\tau)L_\tau^{-1}\,(L_0-L_1)u. \tag{69}$$

Therefore all we need to do is to show the existence of a fixed point of the mapping (from X to X):

$$u\mapsto Tu\equiv L_\tau^{-1}f+(t-\tau)L_\tau^{-1}(L_0-L_1)u. \tag{70}$$

It is clear that if we take $t-\tau$ small enough, we can find $0<r<1$, such that

$$\|Tu-Tv\|_X\leqslant r\,\|u-v\|_X. \tag{71}$$

6. For example, in the case $L=\sum_{i,j}a^{ij}\frac{\partial^2}{\partial x_i\,\partial x_j}$ with $a^{ij}\in C^\alpha$, $X=C^{2,\alpha}$, $Y=C^\alpha$.

Now set $v_0 = 0$ and $v_n = T v_{n-1}$, we see that $\{v_n\}$ is a Cauchy sequence in X and therefore has a limit v which is a fixed point. □

An application of this theorem is to show the existence of the solutions to

$$L' u = \sum_{i,j} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b^i(x) \frac{\partial u}{\partial x_i} + c(x) u(x) = f \tag{72}$$

for Hölder continuous a^{ij}, b^i, c starting from the existence of the Poisson equation which can be shown by explicitly construct the solutions.