Math 527 Lecture 10 Exercises and Problems

Exercises.

Exercise 1. (Evans) Give a direct proof that if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic within a bounded open set Ω, then

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u \tag{1}$$

(Hint: Consider $u_\varepsilon := u + \varepsilon\, |x|^2$, and show that u_ε cannot reach maximum in Ω).

Problems.

Problem 1. Prove

$$u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u \,\mathrm{d}x \text{ for all } B_r(x) \Subset \Omega \Longleftrightarrow u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u \,\mathrm{d}S \text{ for all } B_r(x) \Subset \Omega. \tag{2}$$

Problem 2. (Evans) Modify the proof of the mean-value formulas to show for $n \geqslant 3$ that

$$u(0) = \frac{1}{|\partial B_r|} \int_{\partial B_r} g \,\mathrm{d}S + \frac{1}{n\,(n-2)\,\alpha(n)} \int_{B_r} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f(x) \,\mathrm{d}x \tag{3}$$

if

$$-\triangle u = f \text{ in } B_r, \qquad u = g \text{ on } \partial B_r. \tag{4}$$

Problem 3. Let $v \in C^2(\Omega) \cap C(\bar{\Omega})$ be subharmonic: $-\triangle v \leqslant 0$. Show

a)

$$v(x) \leqslant \frac{1}{|B_r|} \int_{B_r(x)} v \,\mathrm{d}y \tag{5}$$

for all $B_r(x) \subset \Omega$;

b)

$$\max_{\bar{\Omega}} v = \max_{\partial\Omega} v. \tag{6}$$

Problem 4. (Evans) Let Ω be bounded, open. Let u solve

$$-\triangle u = f \text{ in } \Omega; \qquad u = g \text{ on } \partial\Omega \tag{7}$$

Show that there is C depending only on Ω such that

$$\max_{\bar{\Omega}} |u| \leqslant C \left(\max_{\partial\Omega} |g| + \max_{\bar{\Omega}} |f| \right). \tag{8}$$

(Hint: $-\triangle\left(u + \frac{|x|^2}{2\,n} \max |f|\right) \leqslant 0$).

Problem 5. Prove the following Liouville Theorem:

> **Theorem.** *Let u be harmonic and bounded over the whole space $\mathbb{R}^n$. Then u is constant.*

(Hint: Take any $x, y \in \mathbb{R}^n$, apply mean value formulas and try to show that $u(x) - u(y) = 0$)

Problem 6. Prove the following Harnack convergence theorem:

> **Theorem.** *Let u_n be a monotonically increasing sequence of harmonic functions. If there exists $y \in \Omega$ for which $\{u_n(y)\}$ is bounded, then u_n convergens on any subdomain $\Omega' \Subset \Omega$ uniformly and the limit is a harmonic function.*

Problem*.