Math 527 Lecture 9 Exercises and Problems

Exercises.

Exercise 1. Let $\Phi(x)$ be the fundamental solution. Show that $-\triangle\Phi(x)=0$ in the classical sense for all $x\neq 0$.

Problems.

Problem 1. Consider the Poisson equation with Neumann boundary condition in half-space:

$$-\triangle u = f \text{ for } x_n > 0; \qquad \frac{\partial u}{\partial n} = g \text{ for } x_n = 0. \tag{1}$$

Construct the Green's function and represent u.

Problem 2. Obtain the Green's function for Poisson equation with Dirichlet boundary condition for $\Omega = B_1$ the unit ball.

Problem 3. Let $u(x)$ be defined by the Poisson formula for the upper half space $x_n > 0$.

$$u(x) = \frac{2\,x_n}{n\,\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\,\mathrm{d}y \equiv \int_{\partial\mathbb{R}^n_+} K(x,y)\,g(y)\,\mathrm{d}y. \tag{2}$$

Prove that u indeed solves

$$-\triangle u = 0 \text{ for } x_n > 0; \qquad u = g \text{ for } x_n = 0 \tag{3}$$

if we assume g is continuous.

Problem*.

Problem. (Thanks to Mr. Slevinsky) Consider the equation

$$-\triangle u = 0 \text{ in } \Omega \subset \mathbb{R}^2; \qquad u = g \text{ on } \partial\Omega. \tag{4}$$

Prove existence of the solution using the Riemann mapping theorem from Complex Analysis and the fact that u exists when Ω is the unit disk.