Poisson Equations: Explicit Formulas

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Poisson's equation

$$-\triangle u = f. \tag{1}$$

Here $x \in U$, $u: \bar{U} \mapsto \mathbb{R}$, and $U \subset \mathbb{R}^n$ is a given open set. The *Laplacian* $\triangle$ is defined as

$$\triangle u = \sum_{i=1}^{n} u_{x_i x_i}. \tag{2}$$

In general, we need to supplement the above equations with boundary conditions, for example the Dirichlet boundary condition

$$u = g \qquad \text{on } \partial U \tag{3}$$

or the Neumann boundary condition

$$\frac{\partial u}{\partial \boldsymbol{n}} = g \qquad \text{on } \partial U. \tag{4}$$

We notice that the Laplace's equation with nonhomogeneous boundary condition can be transformed into Poisson's equation with homogeneous boundary condition. For example, consider

$$\triangle u = 0 \qquad \text{in } U; \qquad u = g \qquad \text{on } \partial U, \tag{5}$$

Suppose we can extend g (currently only defined on ∂U) to a C^2 function defined on the whole domain U, and denote the result by $\tilde{g}$, then we have

$$-\triangle(u - \tilde{g}) = \triangle \tilde{g} \quad \text{in } U; \qquad u - \tilde{g} = 0 \quad \text{on } \partial U. \tag{6}$$

This shows that when we try to find out the solutions formulas, we should not treat Laplace's and Poisson's equation separately.

In fact, it turns out that the basis for all formulas for the solutions is the formula solving Poisson's equation in the whole space:

$$-\triangle u = f \qquad \text{in } \mathbb{R}^n. \tag{7}$$

Or more precisely, the formula solving Poisson's equation with a special right hand side:

$$-\triangle u = \delta(x) \tag{8}$$

where $\delta(x)$ is the Dirac delta function.

1. Fundamental solution.

We start by trying to solve

$$-\triangle u = \delta(x). \tag{9}$$

1.1. The Dirac delta function.

First we explain the rationale behind this strategy. That is, why solving this equation can give us a formula for the general Poisson's equation with right hand side $f(x)$.

Definition 1. *The Dirac delta function is a non-tradional function which can only be defined by its action on continuous functions:*

$$\int_{\mathbb{R}^n} \delta(x)\, f(x)\, \mathrm{d}x = f(0). \tag{10}$$

Remark 2. It is easy to see that $\delta(x)$ has to satisfy the following:

1. $\delta(x) = 0$ for any $x \neq 0$;
2. $\int_{\mathbb{R}^n} \delta(x) = 1$.

As a consequence, $\delta(x)$ cannot be a function and the integrals involved in the above are not really integrals.

One important property of the above definition is that

$$\int_{\mathbb{R}^n} \delta(x-y)\, f(y)\, \mathrm{d}y = f(x). \tag{11}$$

Recalling the definition of convolution:

$$(f*g)(x) := \int_{\mathbb{R}^n} f(x-y)\, g(y)\, \mathrm{d}y, \tag{12}$$

we can also write the above as

$$\delta * f = f. \tag{13}$$

The most important property of convolution is the following. Let $\mathcal{D}$ be any constant coefficient partial differential operator, then we have

$$\mathcal{D}(f*g) = \mathcal{D}f * g = f * \mathcal{D}g. \tag{14}$$

Thus in particular, if we can solve

$$-\triangle u = \delta \tag{15}$$

and denote its solution by Φ, then for any $f(x)$ we have

$$-\triangle(\Phi * f) = (-\triangle \Phi) * f = \delta * f = f. \tag{16}$$

Therefore, as soon as the obtain $\Phi(x)$, the solution to the general Poisson's equation in the whole space can be written as

$$u(x) = \Phi * f = \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, \mathrm{d}y. \tag{17}$$

We will call this Φ the *fundamental solution*.

1.2. Solving $-\triangle\Phi = \delta$.

It is clear that $\delta(x)$ is radial symmetric, in that its values only depend on the radius $r = |x| = \sqrt{x_1^2 + \cdots + x_n^2}$ but not the direction. Therefore it makes sense to look for Φ which are radial symmetric too.

Set $\Phi = \Phi(r)$. Simple change of variable shows that $-\triangle\Phi$ now becomes

$$-\Phi'' - \frac{n-1}{r}\, \Phi'. \tag{18}$$

To solve $-\triangle\Phi = \delta$, we need to obtain the formula of δ as a function of r only. It may be tempting to guess that the formula is simply $\delta(r)$, but this is wrong.

Lemma 3. *The delta function $\delta(x)$ in $\mathbb{R}^n$, written in polar coordinates, becomes*

$$\Delta(r) = \frac{\delta(r)}{n\,\alpha(n)\, r^{n-1}} \tag{19}$$

where $\alpha(n)$ is the volumne of the n-dimensional unit ball (thus $n\,\alpha(n)$ is the area of the $(n-1)$-dimensional sphere.

Proof. The proof is left as an optional exercise. $\square$

With the help of this lemma, we can write $-\triangle\Phi = \delta$ in Polar coordinates and reduce it to an ODE:

$$-\Phi'' - \frac{n-1}{r}\, \Phi' = \frac{1}{n\,\alpha(n)\, r^{n-1}}\, \delta(r). \tag{20}$$

Multiply both sides by r^{n-1} we have

$$-(r^{n-1}\,\Phi')' = \frac{1}{n\,\alpha(n)}\, \delta(r). \tag{21}$$

Integrate, we obtain

$$-r^{n-1}\,\Phi' = \frac{1}{n\,\alpha(n)} \tag{22}$$

which finally leads to

$$\Phi(r) = \begin{cases} -\frac{1}{2\pi} \log r & n = 2 \\ \frac{1}{n\,(n-2)\,\alpha(n)} \frac{1}{r^{n-2}} & n \geqslant 3 \end{cases}. \tag{23}$$

Definition 4. *The function*

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \log |x| & n = 2 \\ \frac{1}{n\,(n-2)\,\alpha(n)} \frac{1}{|x|^{n-2}} & n \geqslant 3 \end{cases} \tag{24}$$

defined for $x \in \mathbb{R}^n$*,* $x \neq 0$*, is the* fundamental solution of Laplace's equation.

Now we see that the solution to the Poisson's equation

$$-\triangle u = f \tag{25}$$

is

$$u(x) = \int_{\mathbb{R}^n} \Phi(x - y)\, f(y)\, \mathrm{d}y. \tag{26}$$

Now we prove $-\triangle u = f$ for f continuous, and decays fast enough at ∞.

Note that it is only legal to do

$$\frac{\partial}{\partial x} \int f(x, y)\, \mathrm{d}y = \int \frac{\partial}{\partial x} f(x, y)\, \mathrm{d}y \tag{27}$$

when $\frac{\partial}{\partial x} f(x, y)$ is integrable. So we cannot use

$$-\triangle u = \int (-\triangle_x \Phi)(x - y)\, f(y)\, \mathrm{d}y. \tag{28}$$

However it holds that

$$\nabla u = \int (\nabla_x \Phi)(x - y)\, f(y)\, \mathrm{d}y. \tag{29}$$

We use the following property of convolution:

$$D f * g = f * D g = D(f * g) \tag{30}$$

where D is any differential operator with constant coefficients.

So (we use subscript x to denote differentiation w.r.t. x)

$$\begin{aligned} -\triangle u &= -\int (\nabla_x \Phi)(x - y)\, \nabla f(y)\, \mathrm{d}y \\ &= \int (\nabla \Phi)(x - y)\, \nabla f(y)\, \mathrm{d}y \\ &= \lim_{\varepsilon \searrow 0} \int_{|x-y| > \varepsilon} \nabla \Phi(x - y) \cdot \nabla f(y)\, \mathrm{d}y \\ &= \lim_{\varepsilon \searrow 0} \left[-\int_{|x-y| > \varepsilon} \triangle \Phi(x - y)\, f(y)\, \mathrm{d}y + \int_{|x-y| = \varepsilon} \boldsymbol{n} \cdot \nabla \Phi(x - y)\, f(y)\, \mathrm{d}y \right] \end{aligned} \tag{31}$$

Now we can verify that $\triangle \Phi(x - y) = 0$ when $y \neq x$. So the first term vanishes. For the second term, we have (e.g. $n \geqslant 3$. $n = 2$ is similar.)

$$\boldsymbol{n} \cdot \nabla \Phi(x - y) = -\frac{y}{|y|} \frac{1}{n\,(n-2)\,\alpha(n)} (-(n-2)) \frac{1}{|x - y|^{n-1}} \frac{y}{|y|} \tag{32}$$

which is simply

$$\frac{1}{n\,\alpha(n)\,\varepsilon^{n-1}} = \frac{1}{\mathrm{Area}(|x - y| = \varepsilon)} \tag{33}$$

on the integration surface. Now continuity of f gives

$$f(y) = f(x) + o(1) \tag{34}$$

for $|y-x|=\varepsilon$ as $\varepsilon\searrow 0$. Combining the above we see that the limit is actually $f(x)$.

2. Poisson's equation in bounded domain.

Now we move on to the case

$$-\triangle u=f \quad \text{in } U; \qquad u=g \quad \text{on } \partial U. \tag{35}$$

Our first try is to explore to what extend the "naïve" adaptation

$$\int_U \Phi(x-y)\, f(y)\, \mathrm{d}y \tag{36}$$

can work.

Let u be the solution to the equation. Then we have (note that all the differentiation in the following computation is with respect to y)

$$\begin{aligned}\int_U \Phi(x-y)\, f(y)\, \mathrm{d}y &= \int_U \Phi(x-y)\,(-\triangle u)\, \mathrm{d}y \\ &= -\int_U \nabla\cdot[\Phi(x-y)\,\nabla u]+\int_U \nabla\Phi\cdot\nabla u \\ &= -\int_{\partial U}\Phi(x-y)\,\frac{\partial u}{\partial n_y}\,\mathrm{d}S_y+\int_U \nabla\cdot[u\,\nabla\Phi]-\int_U \triangle\Phi\, u \\ &= -\int_{\partial U}\Phi(x-y)\,\frac{\partial u}{\partial n_y}\,\mathrm{d}S_y+\int_{\partial U}\frac{\partial\Phi(x-y)}{\partial n_y}\,u(y)\,\mathrm{d}S_y+u(x).\end{aligned} \tag{37}$$

We see that

$$u(x)=\int_U \Phi(x-y)\, f(y)\, \mathrm{d}y+\int_{\partial U}\Phi(x-y)\,\frac{\partial u}{\partial n_y}\,\mathrm{d}S_y-\int_{\partial U}\frac{\partial\Phi(x-y)}{\partial n_y}\,u(y)\,\mathrm{d}S_y. \tag{38}$$

Since $u=g$ on ∂U, we have

$$u(x)=\int_U \Phi(x-y)\, f(y)\, \mathrm{d}y+\int_{\partial U}\Phi(x-y)\,\frac{\partial u}{\partial n_y}\,\mathrm{d}S_y-\int_{\partial U}\frac{\partial\Phi(x-y)}{\partial n_y}\,g(y)\,\mathrm{d}S_y. \tag{39}$$

Inspecting the above formula, we see that the first and the third terms on the RHS only involve known quantities, while the second one involves $\frac{\partial u}{\partial n_y}$ which cannot be obtained without knowing the solution u itself.

Now observe that, if $\Phi(x-y)=0$ for $y\in\partial U$, then this trouble term vanishes. Of course $\Phi(x-y)\neq 0$ in general. However we can try to modify Φ in the following manner.

First, consider any function

$$G(x,y)=\Phi(x-y)+\phi(x,y) \tag{40}$$

with $\phi(x,y)$ satisfying

$$\triangle_y\phi(x,y)=0 \qquad \text{in } U. \tag{41}$$

Then one can show that

$$u(x)=\int_U G(x,y)\, f(y)\, \mathrm{d}y+\int_{\partial U}G(x,y)\,\frac{\partial u}{\partial n_y}\,\mathrm{d}S_y-\int_{\partial U}\frac{\partial G(x,y)}{\partial n_y}\,g(y)\,\mathrm{d}S_y. \tag{42}$$

But now we can try to choose $\phi(x,y)$ such that

$$\phi(x,y)+\Phi(x-y)=0 \qquad \text{for all } x\in U,\ y\in\partial U. \tag{43}$$

If we are successful, then the function $G(x,y)$ is called the *Green's function* related to the problem

$$-\triangle u=f \qquad \text{in } U; \qquad u=g \qquad \text{on } \partial U. \tag{44}$$

When the explicit formula of the Green's function is available, the solution can be written as

$$u(x)=\int_U G(x,y)\, f(y)\, \mathrm{d}y-\int_{\partial U}\frac{\partial G(x,y)}{\partial n_y}\,g(y)\,\mathrm{d}S_y. \tag{45}$$

In particular, when $f=0$, we have

$$u(x) = -\int_{\partial U} \frac{\partial G(x,y)}{\partial n_y} g(y)\,\mathrm{d}S_y. \tag{46}$$

Remark 5. In general $G(x,y)$ cannot be written as a function of $|x-y|$ anymore. However, one can still show that $G(x,y)=G(y,x)$. See Theorem 13 on p.35 of Evans.

Remark 6. The Green's function always exists. However for general U, an explicit formula of G is either too hard to find or to complicated to be useful.

3. Well-posedness.

How does the above discussion help in the study of well-posedness for the problem $-\triangle u = f$ in U, $u=g$ on ∂U?

- Existence. An explicit formula is given. However the formula involves the Green's function $G(x,y)$ whose formula is not available for all U. Therefore existence is only shown for special domains.
- Uniqueness. As the uniqueness of $G(x,\ y)$ is not established, the above discussion does not say anything about uniqueness.
- Continuous dependence on data. Since this is a stronger requirement than uniqueness, nothing can be said at this stage.

Remark 7. Comparing with last lecture, we see that as the equation gets more complicated, explicit formulas alone yields less information. On the other hand, as we will see later, qualitative theory also has its limitations. A satisfying complete theory is a consequence of a symbiosis of theory and formulas.

4. Poisson's formulas.

One can apply the above theory to the problem

$$\triangle u = 0 \qquad \text{in } U; \qquad u = g \qquad \text{on } \partial U. \tag{47}$$

and get explicit formulas for u when U are special domains. For example,

- U is the half-space

  $$\{(x_1, \ldots, x_n)|\ x_n > 0\}. \tag{48}$$

  We have

  $$u(x) = \frac{2\,x_n}{n\,\alpha(n)} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\,\mathrm{d}y \equiv \int_{\partial \mathbb{R}^n_+} K(x,y)\,g(y)\,\mathrm{d}y. \tag{49}$$

- U is the ball

  $$\{(x_1, \ldots, x_n)|\ x_1^2 + \cdots + x_n^2 < r^2\}. \tag{50}$$

  We have

  $$u(x) = \frac{r^2 - |x|^2}{n\,\alpha(n)\,r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\,\mathrm{d}S(y) \equiv \int_{\partial B(0,r)} K(x,y)\,g(y)\,\mathrm{d}y. \tag{51}$$

These formulas are called *Poisson's formulas*, and the kernels $K(x,y)$ are called *Poisson's kernels.*

See 2.2.4b, 2.2.4c of Evans for details.