Math 527 Lecture 6 Exercises and Problems

Exercises.

Problems.

Problem 1. (Evans) Let E be a closed subset of $\mathbb{R}^n$. Show that if the Hopf-Lax formula could be applied to the initial-value problem

$$\begin{cases} u_t + |Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = \begin{cases} 0 & x \in E \\ +\infty & x \notin E \end{cases} & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{1}$$

it would give teh solution

$$u(t, x) = \frac{1}{4} \operatorname{dist}(x, E)^2. \tag{2}$$

Problem 2. (Evans) Assume u^1, u^2 are two solutions of the initial value problems

$$\begin{cases} u_t^i + H(Du^i) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u^i = g^i & \text{on } \mathbb{R}^n \times \{t = -0\} \end{cases}, \qquad i = 1, 2 \tag{3}$$

given by the Hopf-Lax formula. Prove the L^∞-contraction inequality

$$\sup_{\mathbb{R}^n} |u^1(t, \cdot) - u^2(t, \cdot)| \leqslant \sup_{\mathbb{R}^n} |g^1 - g^2| \tag{4}$$

for all $t > 0$.

Problem*.

Problem. (Evans) Prove that the Hopf-Lax formula reads

$$u(t, x) = \min_{y \in \mathbb{R}^n} \left\{ t\, L\left(\frac{x - y}{t} \right) + g(y) \right\} = \min_{y \in B(x, Rt)} \left\{ t\, L\left(\frac{x - y}{t} \right) + g(y) \right\} \tag{5}$$

for $R = \sup_{\mathbb{R}^n} |DH(Dg)|$, $H = L^*$. **Note that** H is just convex and g is only Lipschitz continuous.