# Math 527 Lecture 6: Hamilton-Jacobi Equation: Explicit Formulas

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**Method of characteristics.**

We try to apply the method of characteristics to the Hamilton-Jacobi equation:

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \tag{1}$$

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \tag{2}$$

To avoid confusion, we use the following notation:

$$\begin{pmatrix} t \\ x \end{pmatrix} \longleftrightarrow \tilde{x}, \quad u \longleftrightarrow z, \quad \begin{pmatrix} u_t \\ Du \end{pmatrix} \longleftrightarrow \tilde{p} = \begin{pmatrix} p_0 \\ p \end{pmatrix}. \tag{3}$$

Then we can re-write the equation to

$$F(\tilde{x}, z, \tilde{p}) = 0 \tag{4}$$

where

$$F(\tilde{x}, z, \tilde{p}) := p_0 + H(x, p). \tag{5}$$

The characteristics ODEs then are

$$\begin{pmatrix} \dot{t} \\ \dot{x} \end{pmatrix} = \dot{\tilde{x}} = D_{\tilde{p}} F = \begin{pmatrix} 1 \\ D_p H \end{pmatrix}, \tag{6}$$

$$\dot{z} = D_{\tilde{p}} F \cdot \tilde{p} = \begin{pmatrix} 1 \\ D_p H \end{pmatrix} \cdot \begin{pmatrix} p_0 \\ p \end{pmatrix} = p_0 + D_p H \cdot p = D_p H \cdot p - H(p, x), \tag{7}$$

$$\begin{pmatrix} \dot{p}_0 \\ \dot{p} \end{pmatrix} = \dot{\tilde{p}} = -(D_z F)\, p - D_x F = -\begin{pmatrix} 0 \\ D_x H \end{pmatrix}. \tag{8}$$

Now we try to solve the characteristic ODEs. First notice that, since $\dot{t} = 1$, we can simply use $t$ as the parameter instead of $s$. Thus the equations become

$$\dot{x} = D_p H, \tag{9}$$

$$\dot{z} = D_p H \cdot p - H(x, p), \tag{10}$$

$$\dot{p} = -D_x H, \tag{11}$$

$$p_0 = p_0|_{t=0}. \tag{12}$$

It is clear the all we need to do is to solve the first 3 equations.

Losing a bit rigor, we assume (for now only) $H$ is differentiable and strictly convex. We also assume $H$ grows super-linearly at infinity:

$$\lim_{|p| \nearrow \infty} \frac{H(x, p)}{|p|} = +\infty, \tag{13}$$

Now take Legendre transform:

$$L(x, v) := \sup_{p \in \mathbb{R}^n} \{v \cdot p - H(x, p)\}. \tag{14}$$

The $z$ equation then becomes

$$\dot{z} = L(x, v) \tag{15}$$

where $q$ satisfies

$$q = D_p H(x, p). \tag{16}$$

Therefore the solution $u$ is given by

$$u(x) = u(x_0) + \int_0^t L(x(\tau), v(\tau))\, \mathrm{d}\tau. \tag{17}$$

where $x$ and $x_0$ are related by

$$\dot{x} = D_p H = v, \qquad x(0) = x_0. \tag{18}$$

To further simplify the system, we notice that

$$\dot{x} = D_p H, \qquad \dot{p} = -D_x H \tag{19}$$

implies

$$-\frac{\mathrm{d}}{\mathrm{d}t}(D_v L) + D_x L = 0 \tag{20}$$

which implies that $q, x$ minimizes

$$\int_0^t L(x(\tau), v(\tau))\,\mathrm{d}\tau \tag{21}$$

with $x(0), x(t)$ fixed.

To see this, write $L(x,v) = v \cdot p(x,v) - H(x, p(x,v))$, and compute

$$D_v L = p + v \cdot D_q p - D_p H \cdot D_v p = p, \tag{22}$$

$$D_x L = v \cdot D_x p - D_x H - D_p H \cdot D_x p = -D_x H, \tag{23}$$

where we have used $v = D_p H$. Now the equation $\dot{p} = -D_x H$ gives what we want.

Thus we see that the Hamilton-Jacobi equation can be solved as soon as we find out the trajectories $x(t)$ and $v(t)$. Below we will see that in a special case, this can indeed be done (in some sense).

**The Hopf-Lax formula.**

This special case is when $H$ is independent of $x$, that is $H = H(Du)$. The characteristic equations can then be further simplified to

$$\begin{aligned} \dot{x} &= D_p H, && (24)\\ \dot{z} &= D_p H \cdot p - H(p) = L(v), && (25)\\ \dot{p} &= D_x H = 0, && (26)\\ p_{n+1} &= p_{n+1}|_{t=0}\,. && (27) \end{aligned}$$

We see that $p$ is a constant vector along the characteristic curve, and as a consequence $\dot{x} = D_p H$ is a constant vector, and therefore the characteristics $x(t)$ are straight lines. Furthermore we know that the velocity $q = \dot{x}$ is constant.

Thus if $x(0) = y$ and $x(t) = x$, we must have

$$v = \frac{x-y}{t}. \tag{28}$$

As a consequence

$$\frac{\mathrm{d}}{\mathrm{d}t} z = L(v) = L\left(\frac{x-y}{t}\right) \Longrightarrow z(t) = z(0) + t\,L\left(\frac{x-y}{t}\right) = g(y) + t\,L\left(\frac{x-y}{t}\right). \tag{29}$$

Now the only problem is that $y$ is not known.

Now think of $g(y)$ as not merely an "initial function", but as an intermediate record. In other words, instead of starting at $t = 0$, imagine our system starts from $t =$, say, $-1$. We consider all possible trajectories emanating from some point at $t = -1$, passing $y$ at $t = 0$, and finally reach time $t$ at $x$. Think of $g(y)$ as the record of work done from $t = -1$ to $t = 0$. Obviously the correct trajectory should be the one that is the minimizer among them all.

**Remark 1.** Note that the above explanation means that the trajectory may not stay $C^1$ as it crosses $t = 0$. This should be expected. Because in general the given $g(y)$ cannot be the result of a dynamical system with $H = H(Du)$, that is "free particle". The $H$ producing $g(y)$ has to be dependent on $x$ or even $u$. Thus there is no surprise that this sudden change leads to a sudden change of direction in the trajectory $v(t)$.

Following this idea, we reach the following Hopf-Lax formula:

$$u(t,x) = z(t) = \inf_{y \in \mathbb{R}^n} \left\{ t\,L\left(\frac{x-y}{t}\right) + g(y) \right\}. \tag{30}$$

**Remark 2.** It can be shown that $L$ grows superlinearly at infinity. As a consequence, if we assume $g$ to be Lipschitz continuous, then the infimum is actually a minimum.

**Remark 3.** Note that convex functions are continuous. The proof can go roughly as follows. First one can show that $f$ (the convex function) is bounded, let the bound be denoted $M$. Then using the definition of convexity we have, for any fixed $x, y$,

$$u(y+\alpha(x-y)) \leqslant u(y)+\alpha\,(u(x)-u(y)) \leqslant u(y)+2\,\alpha\,M. \tag{31}$$

Letting $\alpha \rightarrow 0$ we see that

$$\limsup_{x_n \rightarrow x} u(x_n) \leqslant u(x). \tag{32}$$

On the other hand, for any $x_n \rightarrow x$ we have, by convexity

$$u(x) \leqslant \frac{1}{2}\,[u(x_n)+u(2\,x-x_n)]. \tag{33}$$

This gives

$$u(x) \leqslant \frac{1}{2} \liminf_{x_n \rightarrow x} [u(x_n)+u(2\,x-x_n)]. \tag{34}$$

Continuity then follows.

One can in fact prove that any convex function is Lipschitz continuous, see e.g. B. Dacorogna **Direct Methods in the Calculus of Variations**, 2nd ed., Springer, 2008, §2.3.

**Solution of the H-J equation.**

Now we show that the Hopf-Lax formula

$$u(x,t)=\inf_{y \in \mathbb{R}^n} \left\{t\,L\left(\frac{x-y}{t}\right)+g(y)\right\}. \tag{35}$$

indeed solves the Hamilton-Jacobi equation, albeit only “almost everywhere”.

**Remark 4.** It is easy to see that in general one cannot expect the existence of classical solutions due to possible intersections of characteristics.

There are three things to show.

1. $u=g$ on $\mathbb{R}^n \times \{t=0\}$,
2. $u_t, Du$ exist almost everywhere,
3. $u_t+H(Du)=0$ a.e.

We show them one by one.

1. $u=g$ on $\mathbb{R}^n \times \{t=0\}$.

   Recall the formula:

$$u(t,x)=\min_y \left\{t\,L\left(\frac{x-y}{t}\right)+g(y)\right\}. \tag{36}$$

Taking $y=x$ we have

$$u(t,x) \leqslant g(x)+t\,L(0) \implies \limsup_{t \searrow 0} u(t,x) \leqslant g(x). \tag{37}$$

On the other hand, we compute

$$\begin{aligned} u(t,x) &= \min_y \left\{t\,L\left(\frac{x-y}{t}\right)+g(y)\right\} \\ &= g(x)+\min_y \left\{t\,L\left(\frac{x-y}{t}\right)+g(y)-g(x)\right\} \\ &\geqslant g(x)-\max_y \left\{\mathrm{Lip}(g)\,|y-x|-t\,L\left(\frac{x-y}{t}\right)\right\} \\ &= g(x)-t\max_z \{\mathrm{Lip}(g)\,|z|-L(z)\} \\ &= g(x)-t\max_{w \in B_{\mathrm{Lip}(g)}} \left\{\max_z \{w \cdot z-L(z)\}\right\} \\ &= g(x)-t\max_{w \in B_{\mathrm{Lip}(g)}} H(w). \end{aligned} \tag{38}$$

As $H$ is continuous, we have

$$\liminf_{t\searrow 0} u(t,x) \geqslant g(x). \tag{39}$$

Thus ends the proof.

2. $u_t, Du$ exist almost everywhere.

   It suffices to show that $u$ is Lipschitz with respect to $x$ and to $t$.

   – $u$ is Lipschitz w.r.t. $x$. We estimate $u(t,\hat{x}) - u(t,x)$.

     Choose $y$ such that

     $$u(t,x) = t\,L\Big(\frac{x-y}{t}\Big) + g(y). \tag{40}$$

     Then

     $$u(t,\hat{x}) - u(t,x) \;=\; \min\Big\{t\,L\Big(\frac{\hat{x}-z}{t}\Big) + g(z) - t\,L\Big(\frac{x-y}{t}\Big) - g(y)\Big\}. \tag{41}$$

     Taking $z = \hat{x} - x + y$ (such that $\hat{x} - z = x - y$) we have

     $$u(t,\hat{x}) - u(t,x) \leqslant g(\hat{x} - x + y) - g(y) \leqslant \mathrm{Lip}(g)\,|\hat{x} - x|. \tag{42}$$

     Similarly we can show

     $$u(t,x) - u(t,\hat{x}) \leqslant \mathrm{Lip}(g)\,|\hat{x} - x|. \tag{43}$$

     The Lipschitz continuity of $u$ then follows.

   – $u$ is Lipschitz w.r.t. $t$. This follows from the following property of the Hopf-Lax formula:

     $$u(t,x) = \min_{y\in\mathbb{R}^n}\Big\{(t-s)\,L\Big(\frac{x-y}{t-s}\Big) + u(s,y)\Big\}. \tag{44}$$

     That this should hold is intuitively very clear following our derivation of the formula. For a proof see Evans p. 126.

     Using this formula, we see that estimating $u(t,x) - u(s,x)$ is no different than estimating $u(x,t) - g(x)$. Thus a similar argument as in Step 1. gives

     $$|u(t,x) - u(s,x)| \leqslant C\,|t-s|. \tag{45}$$

3. $u_t + H(Du) = 0$ a.e.

   Fix any $q \in \mathbb{R}^n$, we compute

   $$\begin{aligned} u(t+h, x+h\,q) \;&=\; \min\Big\{h\,L\Big(\frac{x+h\,q-y}{h}\Big) + u(t,y)\Big\} \\ &\leqslant\; h\,L(q) + u(t,x). \end{aligned} \tag{46}$$

   This implies

   $$u_t + q\cdot Du \leqslant L(q) \iff -u_t \geqslant Du\cdot q - L(q) \tag{47}$$

   for all $q \in \mathbb{R}^n$. Therefore

   $$-u_t \geqslant \max_q\{Du\cdot q - L(q)\} = H(Du) \tag{48}$$

   and

   $$u_t + H(Du) \leqslant 0. \tag{49}$$

   For the other direction (that is $u_t + H(Du) \geqslant 0$), we only need to find one $q$ such that

   $$u_t + q\cdot Du \geqslant L(q) \tag{50}$$

   or more specifically

   $$\frac{u(t,x) - u(s,y)}{t-s} \geqslant L(q) \tag{51}$$

   where $x - y$ is in the direction of $q$.

As $u$ is a minimum, to get $u(t, x) - u(s, y) \geqslant$ something, we get rid of the minimum in $u(t, x)$. Take $z$ such that

$$u(t, x) = t\, L\Big(\frac{x - z}{t}\Big) + g(z). \tag{52}$$

Now that $q = \frac{x - z}{t}$ is already chosen, $y$ has to be on the line segment connecting $x$ and $z$. Thus we take

$$s = t - h, \quad y = \frac{s}{t}\, x + \Big(1 - \frac{s}{t}\Big)\, z. \tag{53}$$

Then we have

$$\frac{x - z}{t} = \frac{y - z}{s} = q \tag{54}$$

This gives

$$\begin{aligned} u(t, x) - u(s, y) &\geqslant t\, L\Big(\frac{x - z}{t}\Big) + g(z) - \Big[s\, L\Big(\frac{y - z}{s}\Big) + g(z)\Big] \\ &= (t - s)\, L\Big(\frac{x - z}{t}\Big). \end{aligned} \tag{55}$$

As

$$\frac{u(t, x) - u(s, y)}{t - s} \longrightarrow u_t + \frac{x - z}{t} \cdot D u \tag{56}$$

we get

$$u_t + \frac{x - z}{t} \cdot D u \geqslant L\Big(\frac{x - z}{t}\Big) \tag{57}$$

and finishes the proof.