Math 527 Lecture 5 Exercises and Problems

Exercises.

Exercise 1. Solve the eikonal equation

$$|Du| = 1, \qquad u = 0 \text{ along } x_n = 0. \tag{1}$$

using method of characteristics.

Exercise 2. (Evans) Compute the Legendre transform of

a) $H(p) = \frac{1}{r}|p|^r$ for $1 < r < \infty$. Here $p \in \mathbb{R}^n$ and $|p| := \sqrt{p_1^2 + \cdots + p_n^2}$. (Ans. $\frac{1}{s}|v|^s$, with $1/s + 1/r = 1$)

b) $H(p) = \frac{1}{2}\sum_{i,j=1}^n a_{ij} p_i p_j + \sum_{i=1}^n b_i p_i$. Here $A = (a_{ij})$ is a symmetric and positive definite matrix, $b \in \mathbb{R}^n$.

Problems.

Problem 1. (Evans) Let $H: \mathbb{R}^n \mapsto \mathbb{R}$ be convex. We say v belongs to the subdifferential of H at p, written

$$v \in \partial H(p), \tag{2}$$

if

$$H(r) \geqslant H(p) + v \cdot (r - p) \tag{3}$$

for all $r \in \mathbb{R}^n$. Prove $v \in \partial H(p)$ if and only if $p \in \partial L(v)$ if and only if $p \cdot v = H(p) + L(v)$, where $L = H^*$.

Problem 2. (Evans) Assume L_1, L_2: $\mathbb{R}^n \mapsto \mathbb{R}$ are convex, smooth and superlinear (that is $L/|x| \to \infty$ as $|x| \to \infty$). Show that

$$\min_{v \in \mathbb{R}^n} (L_1(v) + L_2(v)) = \max_{p \in \mathbb{R}^n} (-H_1(p) - H_2(-p)) \tag{4}$$

where $H_1 = L_1^*, H_2 = L_2^*$.

Problem 3. Show that method of characteristics indeed gives the solution to $F(x, u, Du) = 0$.

Problem*.