Math 527 Lecture 5: Method of Characteristics for Fully Nonlinear 1st Order Equations

Sep. 21, 2011

Physical Motivation.

Example 1. (Geometric Optics) Consider the problem of wave (light, sound, etc.) propagation. Let $T(x, y, z)$ be the time the wave reaches the point (x, y, z). We will derive a PDE for it.

Let $c(x, y, z)$ denote the speed of propagation of this particular wave at point (x, y, z). We have

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} + c(x, y, z)\, \boldsymbol{n}\, \delta t\right) = T(x, y, z) + \delta t + o(\delta t). \tag{1}$$

From this we easily get

$$c(x, y, z)\, \nabla T \cdot \boldsymbol{n} = 1. \tag{2}$$

But we notice that $\boldsymbol{n} \parallel \nabla T$ so the above becomes the eikonal equation

$$|\nabla T| = \frac{1}{c(x, y, z)}. \tag{3}$$

Example 2. Recall the theory of Lagrangian and Hamiltonian mechanics.

- Lagrangian mechanics.

  The evolution of a mechanical system is governed by the following principle: There is a function $L(t, x, \dot{x})$, of time, location, and velocity, such that the actual path a particle takes is critical for the functional

  $$I(x, \dot{x}; t, x_{\text{start}}, x_{\text{end}}) := \int_0^t L(s, x(s), \dot{x}(s))\, \mathrm{d}s, \qquad x(0) = x_{\text{start}},\ x(t) = x_{\text{end}}. \tag{4}$$

  meaning that if we perturb x a little bit, to a new path $x + \delta x$ (with starting and end points fixed) then the change in I is of higher order. More specifically, we need

  $$I\big(x + \delta x, \dot{x} + \dot{\delta x}\big) - I(x, \dot{x}) = o(\delta x). \tag{5}$$

  Calculating the left hand side:

  $$\begin{aligned} I\big(x + \delta x, \dot{x} + \dot{\delta x}\big) - I(x, \dot{x}) &= \int_0^t L\big(s, x + \delta x, \dot{x} + \dot{\delta x}\big) - L(s, x, \dot{x})\, \mathrm{d}s \\ &= \int_0^t \frac{\partial L}{\partial x}\, \delta x + \frac{\partial L}{\partial \dot{x}}\, \dot{\delta x}\, \mathrm{d}s + o(\delta x) \\ &= \int_0^t \left[\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\right] \delta x\, \mathrm{d}t + o(\delta x). \end{aligned} \tag{6}$$

  Now it is clear that the condition is

  $$\int_0^t \left[\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\right] \delta x\, \mathrm{d}t = 0 \tag{7}$$

  for any perturbation δx. This gives the Euler-Lagrange equation

  $$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \tag{8}$$

  which has to be satisfied by any path a real particle follows.

- Hamiltonian mechanics.

We de-couple x and $\dot{x}$ and consider L as a function of three independent variables $L = L(t, x, q)$. A change of perspective follows from introducing a new variable:

$$p := \frac{\partial L}{\partial q}. \tag{9}$$

The correspondence $p \leftrightarrow q$ is one-to-one when L satisfies certain conditions, such as L is convex with respect to q (in 1D, this condition becomes $\frac{\partial L}{\partial q}$ is monotone with respect to $\dot{x}$).

We further introduce a new function $H(t, x, p)$ through

$$H(t, x, p)|_{p = \frac{\partial L}{\partial q}} = \frac{\partial L}{\partial q} \cdot q - L(t, x, q). \tag{10}$$

In other words

$$H(t, x, p) = \sup_{q \in \mathbb{R}^n} (p \cdot q - L(t, x, q)). \tag{11}$$

Remark 3. H is the so-called "Legendre transform" of L.

Consider a function $f(q)$. Its Legendre transform $f^*(p)$ is defined as

$$f^*(p) = \sup_{q \in \mathbb{R}^n} (p \cdot q - f(q)). \tag{12}$$

Note that at the minimizer we necessarily have

$$p = \nabla_q f. \tag{13}$$

Some important properties of Legendre transforms:

- f^* is always convex.
- $f^{**} \geqslant g$ for any convex $g \leqslant f$. In particular, if f is convex, then $f^{**} = f$.
- By definition we have the following more symmetric relation:

$$f^*(p) + f(q) \geqslant p \cdot q \tag{14}$$

  for any p, q.

Back to H and L. When L is convex, we have

$$L = H^* \tag{15}$$

which gives

$$q = \frac{\partial H}{\partial p}. \tag{16}$$

Now consider the case when $x, \dot{x}$ is the actual location and velocity of a particle. We know that the Euler-Lagrange equation is satisfied:

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \tag{17}$$

Now we try to write the same thing in H, x, and p.[1]

First $q = \frac{\partial H}{\partial p}$ immediately gives

$$\dot{x} = \frac{\partial H}{\partial p}. \tag{18}$$

Next using $p = \frac{\partial L}{\partial \dot{x}}$ and the Euler-Lagrange equation we get

$$\dot{p} = \frac{\partial L}{\partial x} \tag{19}$$

1. The physical meaning of p is momentum. For example, for a free particle $L = \frac{m\,v^2}{2}$ which gives $p = m\,v$ (here we use the traditional notation v in place of q).

To write this in H we recall the definition

$$H(t,x,p)=p\cdot q-L(t,x,q). \tag{20}$$

It is beneficial to write $q=q(t,x,p)$ to get all dependence clear. When we do this, we have

$$\frac{\partial H}{\partial x}=p\cdot\frac{\partial q}{\partial x}-\frac{\partial L}{\partial x}-\frac{\partial L}{\partial q}\cdot\frac{\partial q}{\partial x}=-\frac{\partial L}{\partial x}. \tag{21}$$

Thus we get the Hamiltonian formulation

$$\dot{x}=\frac{\partial H}{\partial p};\qquad \dot{p}=-\frac{\partial H}{\partial x}. \tag{22}$$

- Hamilton-Jacobi equations.

  Consider the action function:

$$u(t,z;z_0):=\int_0^t L(s,x(s),\dot{x}(s))\,\mathrm{d}s \tag{23}$$

where $(x,\dot{x})$ is the actual path a particle takes (that is the Euler-Lagrange equation, or equivalently the Hamiltonian equation, is satisfied), and further satisfy $x(0)=z_0$, $x(t)=z$. We will derive the equation for u.

First consider

$$u(t+\delta t,z+\dot{x}(t)\,\delta t;z_0)=\int_0^{t+\delta t} L(s,x(s),\dot{x}(s))\,\mathrm{d}s+o(\delta x) \tag{24}$$

We easily obtain

$$\frac{\partial u}{\partial t}+\dot{x}\cdot D_z u=L(t,z,\dot{x}) \tag{25}$$

which gives

$$\frac{\partial u}{\partial t}+[(D_z u)\cdot\dot{x}-L(t,z,\dot{x})]=0. \tag{26}$$

Next we try to figure out what $D_z u$ is:

$$\begin{aligned} u(t,z+\delta z;z_0)-u(t,z;z_0) &= \int_0^t \left[L\big(s,x+\delta x,\dot{x}+\dot{\delta x}\big)-L(s,x,\dot{x})\right]\mathrm{d}s \\ &= \int_0^t \left[\frac{\partial L}{\partial x}\cdot\delta x+\frac{\partial L}{\partial \dot{x}}\cdot\dot{\delta x}\right]\mathrm{d}s+o(\delta z) \\ &= \int_0^t \frac{\partial L}{\partial x}\cdot\delta x\,\mathrm{d}s+\frac{\partial L}{\partial \dot{x}}\cdot\delta x|_0^t-\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x\,\mathrm{d}s+o(\delta z) \\ &= \frac{\partial L}{\partial \dot{x}}\cdot\delta z+\int_0^t\int_0^t\left[\frac{\partial L}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\delta x\,\mathrm{d}s+o(\delta z). \end{aligned} \tag{27}$$

The second term is 0 following the Euler-Lagrange equation. So we see that

$$D_z u=\frac{\partial L}{\partial \dot{x}}. \tag{28}$$

This gives

$$\frac{\partial u}{\partial t}+\left[\frac{\partial L}{\partial \dot{x}}\cdot\dot{x}-L(t,z,\dot{x})\right]=0 \tag{29}$$

which is simply (back to the usual notation x)

$$\frac{\partial u}{\partial t}+H(t,x,D_x u)=0. \tag{30}$$

Note that H is usually nonlinear. So this is a fully nonlinear equation.

- In our class we will only consider the simple case

$$\frac{\partial u}{\partial t}+H(x,D_x u)=0. \tag{31}$$

or even the simplest

$$\frac{\partial u}{\partial t} + H(D_x u) = 0. \tag{32}$$

Note that the motion of a free particle belongs to the last case. In these simple cases we will drop the subscript x and simply write

$$\frac{\partial u}{\partial t} + H(x, Du) = 0. \tag{33}$$

Method of Characteristics for Fully Nonlinear 1st Order Equations.

- Consider the general first order equation

$$F(x, u, Du) = 0 \qquad \text{in } U; \qquad u = g \quad \text{on } \Gamma \subseteq \partial U. \tag{34}$$

Recall that, when F is quasi-linear,

$$F(x, u, Du) = b(x, u) \cdot Du - f(x, u), \tag{35}$$

we can try to solve the ODE system:

$$\dot{x}(s) = b(x, z) \tag{36}$$

$$\dot{z}(s) = f(x, z) \tag{37}$$

and then try to represent s using x and finally obtain the solution from

$$u(x) = z(s). \tag{38}$$

This approach does not work anymore when F is fully nonlinear. For example, when

$$F(x, u, Du) = |Du|^2 - f(x, u), \tag{39}$$

the above approach would give

$$\dot{x}(s) = Du(x) \tag{40}$$

$$\dot{z}(s) = f(x, u). \tag{41}$$

As Du is not known, the ODE system cannot be solved.

Nevertheless, the main idea: reduce the PDE to a system of ODEs along particular curves $x(s)$, still works.

- Consider a curve $x(s)$ to be determined. We try to find out whether it is possible to simplify the PDE along this particular curve.

Clearly $x(s)$ and $z(s) := u(x(s))$ are unknown functions that need to be solved. How about Du? If we only consider the function along $x(s)$, then Du and u has no relation except

$$\frac{\mathrm{d}}{\mathrm{d}s} u = \dot{x}(s) \cdot Du. \tag{42}$$

Therefore $p(s) := Du(x(s))$ should also be treated as an independent unknown function.

Now we derive equations for x, z, p.

First, from the above discussion,

$$\dot{z}(s) = \dot{x}(s) \cdot p(s). \tag{43}$$

To figure out the equation for x, we need to first look at the equation for p, as x will be chosen to make other equations as simple as possible.

Compute

$$\dot{p}(s) = \frac{\mathrm{d}}{\mathrm{d}s} Du(x(s)) = D^2 u(x(s)) \cdot \dot{x}(s). \tag{44}$$

The quantity $D^2 u$ is not known and has to be cancelled.

To cancel it, we turn to the equation. Differentiating $F(x, u, Du) = 0$ we obtain

$$D^2 u \cdot D_p F + (D_z F)\, Du + D_x F = 0 \iff D^2 u \cdot D_p F = -(D_z F)\, Du - D_x F. \tag{45}$$

We see that the D^2u in the p equation can be cancelled provided we require

$$\dot{x}(s) = D_p F. \tag{46}$$

Thus the ODE system we look for is

$$\begin{aligned} \dot{x} &= D_p F, & (47)\\ \dot{z} &= \dot{x}\cdot p = D_p F\cdot p, & (48)\\ \dot{p} &= -(D_z F)\, p - D_x F. & (49) \end{aligned}$$

We see that the system is closed, that is, does not involve any unknown quantities except x, z, p.

Example 4. When F is quasi-linear, that is

$$F(x,z,p) = b(x,z)\cdot p - f(x,z), \tag{50}$$

we have

$$\begin{aligned} \dot{x} &= b(x,z), & (51)\\ \dot{z} &= b(x,z)\cdot p = f(x,z), & (52)\\ \dot{p} &= -(D_z b\cdot p - D_z f)\, p - (D_x b\cdot p - D_x f). & (53) \end{aligned}$$

We see that the first two equations and the third equation are decoupled. Thus we can solve the first two equations and obtain the solution. There is no need to solve the 3rd equation.

Summarizing, we have shown that if u solves $F(Du, u, x) = 0$, and $\dot{x}(s) = D_p F(Du, u, x)$, then $p(s) := Du(x(s))$, $z(s) := u(x(s))$ solves

$$\dot{z} = D_p F\cdot p, \qquad \dot{p} = -(D_z F)\, p - D_x F. \tag{54}$$

Obviously the opposite direction is more important: Suppose we solve the characteristic ODE system, does the solution gives us the solution to the original equation?

Example 5. (Hamilton-Jacobi equation) Consider

$$u_t + H(Du, x) = 0. \tag{55}$$

Let

$$F(p,z,x) = p_{n+1} + H\left(\begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right). \tag{56}$$

Then we have

$$\begin{aligned} \dot{x} &= D_p F = \begin{pmatrix} D_p H \\ 1 \end{pmatrix} \Longleftrightarrow \dot{x} = D_p H, \quad \dot{t} = 1. & (57)\\ \dot{z} &= D_p F\cdot p = p_{n+1} + D_p H\cdot p = D_p H\cdot p - H, & (58)\\ \dot{p} &= -(D_z F)\, p - D_x F = \begin{pmatrix} -D_x H \\ 0 \end{pmatrix} \Longleftrightarrow \dot{p} = -D_x H, \quad \dot{p}_{n+1} = 0. & (59) \end{aligned}$$

Note that in the above the notation is a bit messy, as x is used for both the spatial variable x and $\begin{pmatrix} x \\ t \end{pmatrix}$, and p is used for both Du and $\begin{pmatrix} Du \\ u_t \end{pmatrix}$.

As $\dot{t} = 1$, we can replace s by t. Thus the characteristics equations become

$$\dot{x} = D_p H, \quad \dot{z} = \cdots, \quad \dot{p} = -D_x H, \quad \dot{p}_{n+1} = 0. \tag{60}$$

Now we recognize that the x, p equations are

$$\dot{x} = D_p H, \quad \dot{p} = -D_x H \tag{61}$$

which are the so-called Hamiltonian equations which governs the evolution of particles (x is the location and p is the momentum of the particle). This understanding is important in deriving the solution formula for the H-J equation.