Math 527 Lecture 4 Exercises and Problems

Exercises.

Exercise 1. The most "natural" finite difference discretization of the equation is obviously

$$\frac{u_n^{k+1}-u_n^k}{\Delta t}+\frac{f(u_{n+1}^k)-f(u_n^k)}{\Delta x}=0. \tag{1}$$

Try to carry out the existence proof using this scheme. At which step does it break down?

Problems.

Problem 1. Recall

Theorem 1. *Let $u_0 \in L^\infty(\mathbb{R})$, and let $f \in C^2(\mathbb{R})$ with $f''>0$ on $\{u\colon |u| \leqslant \|u_0\|_{L^\infty}\}$. Then there exists a weak solution u with the following properties:*

a) $|u(x,t)| \leqslant \|u_0\|_{L^\infty} \equiv M$, $(x,t) \in \mathbb{R}\times\mathbb{R}^+$.

b) There is a constant $E>0$, depending only on M, $\mu=\min\{f''(u)\colon |u| \leqslant \|u_0\|_{L^\infty}\}$ and $A=\max\{|f'(u)|\colon |u| \leqslant \|u_0\|_{L^\infty}\}$ such that for every $a>0, t>0$, and $x\in\mathbb{R}$,

$$\frac{u(x+a,t)-u(x,t)}{a}<\frac{E}{t}. \tag{2}$$

c) u is stable and depends continuously on u_0 in the following sense: If $v_0 \in L^\infty(\mathbb{R})$ with $\|v_0\|_{L^\infty} \leqslant \|u_0\|_{L^\infty}$, and v is the corresponding solution constructed from the process in the proof, then for every $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$, and every $t>0$,

$$\int_{x_1}^{x_2} |u(x,t)-v(x,t)|\,\mathrm{d}x \leqslant \int_{x_1-At}^{x_2+At} |u_0(x)-v_0(x)|\,\mathrm{d}x. \tag{3}$$

Now let u be a piecewise smooth weak solution with compact support satisfying the naiver entropy condition $u(t,x+) \leqslant u(t,x-)$. Show that it satisfies a) – c).

Problem 2. Let $f\colon \mathbb{R}\mapsto\mathbb{R}$ be of locally bounded total variation, that is the total variation over any finite interval is finite. Prove that

1. f can have at most countably many jump discontinuities. (Hint: Consider the number of jumps of size $>1/n$ over the interval $[-n,n]$ and then sum up.)
2. f is differentiable almost everywhere. (Hint: Define $g(x)=$ The total variation of f over $[-x,x]$, show that both g and $f-g$ are monotone. The fact that monotone functions are differentiable almost everywhere can be taken for granted.)

Problem*.