Math 527 Lecture 3 Exercises and Problems

Exercises.

Exercise 1. (Evans) Show that

$$u(t,x)=\begin{cases}-\frac{2}{3}\left(t+\sqrt{3x+t^2}\right) & 4x+t^2>0\\ 0 & 4x+t^2<0\end{cases}\tag{1}$$

is an (unbounded) entropy solution of $u_t+\left(\frac{u^2}{2}\right)_x=0$.

Exercise 2. Construct entropy solutions for the following initial value problems

a)

$$u_t+\left(\frac{u^2}{2}\right)_x=0, \qquad u(0,x)=\begin{cases}1 & x<-2\\ 0 & x>3\end{cases}\tag{2}$$

b)

$$u_t+\left(\frac{u^2}{2}\right)_x=0, \qquad u(0,x)=\begin{cases}2 & x<0\\ 1 & 0<x<1\\ 0 & x>1\end{cases}\tag{3}$$

c)

$$u_t+\left(\frac{u^4}{4}\right)x=0, \qquad u(0,x)=\begin{cases}1 & x<0\\ 0 & x>0\end{cases}.\tag{4}$$

Exercise 3. Consider the following problem

$$u_t+\left(\frac{u^2}{2}\right)_x=0, \qquad u(0,x)=\begin{cases}1 & x<-\varepsilon\\ \frac{\varepsilon-x}{2\varepsilon} & -\varepsilon<x<\varepsilon\\ 0 & x>\varepsilon\end{cases}\tag{5}$$

Construct the entropy solution and study what happens when $\varepsilon\searrow 0$.

Exercise 4. (Irreversibility) Consider the Burgers equation

$$u_t+\left(\frac{u^2}{2}\right)_x=0, \qquad u(0,x)=g(x).\tag{6}$$

Consider the following two initial values

$$g_1=\begin{cases}1 & x<1/2\\ 0 & x>1/2\end{cases}; \qquad g_2(x)=\begin{cases}1 & x<0\\ 1-x & 0<x<1\\ 0 & x>1\end{cases}.\tag{7}$$

Let u_1, u_2 be the entropy solutions starting from g_1, g_2 respectively. Show that $u_1=u_2$ for all $t>1$. Thus there is "irreversibility" as information is "lost" as solutions evolve.

Problems.

Problem 1. (Evans) Compute explicitly the unique entropy solution of

$$u_t+\left(\frac{u^2}{2}\right)_x=0, \qquad u(0,x)=g\tag{8}$$

for

$$g(x)=\begin{cases}1 & x<-1\\ 0 & -1<x<0\\ 2 & 0<x<1\\ 0 & x>1\end{cases}.\tag{9}$$

Draw a picture of your answer. Be sure to illustrate what happens for all times $t>0$.

Problem 2. Solve the Burgers equation

$$u_t+\left(\frac{u^2}{2}\right)_x=0\tag{10}$$

with initial condition $g(x)=\sin x+1$.

Problem 3. Consider 1D scalar conservation law

$$u_t+f(u)_x=0\tag{11}$$

with f concave (that is $f''<0$). Develop a complete theory of entropy solutions for such problem.

Problem 4. (Irreversibility) Consider a general 1D scalar conservation law

$$u_t+f(u)_x=0, \qquad u(0,x)=u_0(x).\tag{12}$$

Show that for any convex f (that is $f'' > 0$ everywhere), we can find two different initial values u_{01} and u_{02}, and then determine a time $T > 0$, such that the solutions u_1 and u_2 coincide for $t > T$.

Problem*.