Math 527 Lecture 3: Conservation Laws II

Sep. 14, 2011

Construction of weak solutions for simple initial data.

- Problem with method of characteristics:
  1. Crossing; For example
  $$u_t + u\,u_x = 0, \qquad u(0,x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases} \tag{1}$$
  2. Void. For example
  $$u_t + u\,u_x = 0, \qquad u(0,x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \tag{2}$$

- How does "weak solution" resolve these two issues?
  - Crossing.

    Introduce a curve of "jump discontinuity". This curve has to satisfy the jump condition:
    $$\frac{\mathrm{d}x}{\mathrm{d}t} = s(x,t) = \frac{f(u_l) - f(u_r)}{u_l - u_r}. \tag{3}$$
  - Note that we should only use "jump discontinuity" to resolve crossing. If we use it to deal with void, we would get unreasonable solutions.

    Example 1. Consider
    $$u_t + u\,u_x = 0, \qquad u(0,x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \tag{4}$$
    We can define
    $$u(t,x) = \begin{cases} 0 & x < t/2 \\ 1 & x > t/2 \end{cases}. \tag{5}$$
    The jump condition is satisfied therefore u is indeed a weak solution.

    However we claim that it is not reasonable. To see this, consider the following problem
    $$u_t + u\,u_x = 0, \qquad u(0,x) = \begin{cases} 0 & x < -\varepsilon \\ \dfrac{x+\varepsilon}{2\,\varepsilon} & -\varepsilon < x < \varepsilon \\ 1 & x > 0 \end{cases} \tag{6}$$
    We can clearly see that its solution approaches
    $$u(t,x) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < t \\ 1 & x > t \end{cases}. \tag{7}$$
    Therefore using discontinuities to fill "voids" should not be allowed.
  - This leads to the idea of "entropy solution". An entropy solution is a weak solution which further satisfies an "entropy condition" which excludes unreasonable solutions like the above. For 1D scalar equation, we can design the following
    - Toy entropy condition: For convex $f(u)$, a weak solution should only have discontinuities satisfying $u_l > u_r$.
    - This entropy condition is not useful in proofs due to the following reason. This condition is only meaningful when u is "piecewise C^1". However, in practice such information is usually very hard to establish and is often not true.

- A better one that is meaningful for any L^∞ function is the following:

$$\frac{u(x+a,t)-u(x,t)}{a} \leqslant \frac{E}{t} \tag{8}$$

for all $a>0$, $t>0$.

∘ Void.

Guided by the above example, we try to fill "voids" using simple wave solution as

$$u(t,x)=U\left(\frac{x-x_0}{t-t_0}\right). \tag{9}$$

Substituting into the equation

$$u_t + a(u)\, u_x = 0 \tag{10}$$

we obtain

$$a(U(\xi)) = \xi. \tag{11}$$

In particular, for Burgers equation we have

$$u(t,x)=U\left(\frac{x-x_0}{t-t_0}\right)=\frac{x-x_0}{t-t_0}. \tag{12}$$

Here (x_0, t_0) is the "tip" of the wedge-like void region.

Example 2. [1]Construct the entropy solution for (Sorry guys, how come I copied $u=0$ for $|x|>1$?!)

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \qquad u(0,x) = \begin{cases} 1 & |x|>1 \\ -1+|x| & |x|<1 \end{cases}. \tag{13}$$

Solution. First we have to understand the structure of characteristics.

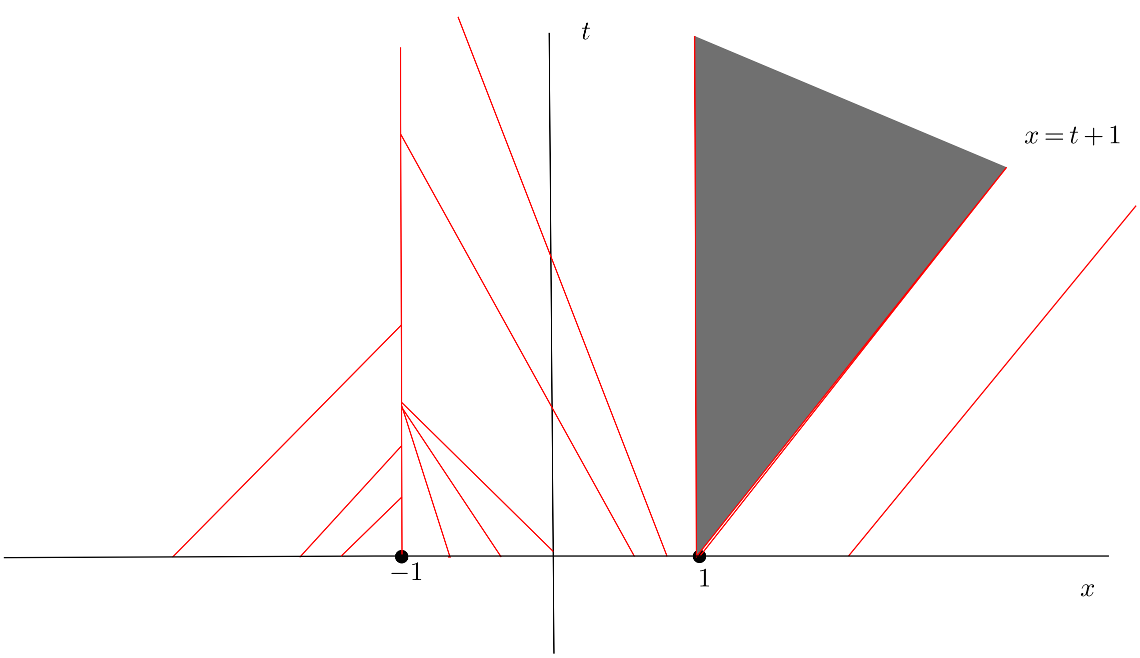

Figure 1. Structure of characteristics

1. Taken from J. Kevorkian, "Partial Differential Equations: Analytical Solution Techniques".

From this picture it is clear that

1. Right of $x=t+1$, $u=1$;
2. Left of $x=-1$, $u=1$.
3. There is no characteristics in the gray area;
4. There are lots of crossings for $x<1$. In particular, the very first crossing occurs at $x=-1$ at $t=0$.

Based on the above understanding, we try to construct the solution as follows:

1. $u=1$ for $x>t+1$;
2. A rarefaction wave $\frac{x-1}{t}$ filling the gray area;
3. A shock emanating from the point $x=-1, t=0$. To the right of the shock, $u=1$. To the left, using the fact that u is constant along the straight lines, we see that $u=\frac{x+1}{t-1}$ below $x=-t$, and $u=\frac{x-1}{t+1}$ above it.

Now we determine the shock speed:

- Before meeting $x=-t$, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t}=s=\frac{\left(\frac{x+1}{t-1}\right)^2-1}{\frac{x+1}{t-1}-1}=\frac{\frac{x+1}{t-1}+1}{2}=\frac{x+t}{2\,(t-1)}, \qquad x(0)=-1 \tag{14}$$

  This is a linear equation and can be solve to get

$$x=-2+t+\sqrt{1-t} \tag{15}$$

- We compute the point of meeting (note that t needs to be positive):

$$-t=-2+t+\sqrt{1-t}\Longrightarrow t=3/4. \tag{16}$$

  So the shock is given by $x=-2+t+\sqrt{1-t}$ from $x=-1, t=0$ to $x=-3/4, t=3/4$.

- The 2nd part of the shock is determined through solving

$$\frac{\mathrm{d}x}{\mathrm{d}t}=s=\frac{\frac{x-1}{t+1}+1}{2}=\frac{x+t}{2\,(t+1)}, \qquad x(3/4)=-3/4. \tag{17}$$

  The solution is given by

$$x(t)=2+t-\sqrt{7\,(1+t)}. \tag{18}$$

- At this point it is clear that the shock will ultimately meet $x=1$, and then "invading" into the gray area. To compute this 3rd part of the shock, we first get the "meeting point":

$$1=2+t-\sqrt{7\,(1+t)}\Longrightarrow t=6. \tag{19}$$

- The shock from the point $x=1, t=6$ is governed by

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\frac{x-1}{t}+1}{2}, \qquad x(6)=1 \tag{20}$$

  whose solution is

$$x(t)=1+t-\sqrt{6\,t}. \tag{21}$$

- As $x(t)=1+t-\sqrt{6\,t}<1+t$ for all t, the shock will never meet $x=1+t$, the boundary of the right side.

Remark 3. Note that as $t\to\infty$, the solution approaches a constant $u\equiv 1$. In other words, all the "details" between $-1<x<1$ are lost.