Math 527 Lecture 2 Exercises and Problems

Exercises.

Exercise 1. (Evans) Confirm that the formula $u = g(x - t\, f'(u))$ provides an implicit solution for the conservation law

$$u_t + f(u)_x = 0. \tag{1}$$

Exercise 2. (Evans) Assume $f(0) = 0$, u is a continuous integral solution of the conservation law

$$u_t + f(u)_x = 0, \qquad u(0, x) = g \tag{2}$$

and u has compact support in $\mathbb{R} \times [0, T]$ for each $T > 0$ (meaning: for each $T > 0$ there is R – may depend on T – such that $u = 0$ outside $[-R, R] \times [0, T]$. Prove

$$\int_{-\infty}^{\infty} u(t, x)\, \mathrm{d}x = \int_{-\infty}^{\infty} g(x)\, \mathrm{d}x. \tag{3}$$

for all $t > 0$.

Exercise 3. Show that

$$u(t, x) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < t \\ 1 & x > t \end{cases} \tag{4}$$

is a weak solution to the problem

$$u_t + u\, u_x = 0, \qquad u(0, x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}. \tag{5}$$

Exercise 4. Consider the 1D scalar conservation law

$$u_t + f(u)_x = 0, \qquad u(0, x) = g(x). \tag{6}$$

Let $u \in C^1$ be a weak solution. Show that u is also a classical solution, that is satisfy the above in the classical pointwise sense.

Problems.

Problem 1. Consider the 1D scalar conservation law

$$u_t + f(u)_x = 0, \qquad u(0, x) = g(x). \tag{7}$$

Let u be piecewise C^1, with possible jumps along finitely many curves in the x-t plane. Further assume that

1. u satisfies $u_t + f(u)_x = 0$ pointwise away from the jumps.
2. u satisfies the jump conditions.
3. $u(t, x) \to g(x)$ uniformly as $t \searrow 0$.

Show that u is a weak solution.

Problem 2. (Traffic Flow) Consider traffic on a one-lane road with no on and off ramps. Let ρ denote that traffic density.

a) Derive a simple conservation law model for the evolution of ρ.

b) Discuss: What are the appropriate assumptions on the flux function?

c) Try to solve your traffice model using tools built in this lecture.

Problem 3. (General Quasilinear 1st Order PDE)

a) Establish the solution strategy of the general quasi-linear 1st order PDE in 2D

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u), \qquad u(x_0(s), y_0(s)) = u_0(s) \tag{8}$$

using the method of characteristics. Here a, b, c are as smooth as you want, and $(x_0(s), y_0(s))$ is a smooth curve in the x-y plane.

b) Discuss (proofs not necessary) theoretical issues involved.

c) Apply your method to the following problem

$$u\, u_x + u_y = 1, \qquad u(s, s) = 2\, s. \tag{9}$$

Problem 4. (Quasilinear 1st Order PDE in Higher Dimensions) Solve the problem

$$u_x + u\, (u_y + u_z) = 0, \qquad u(0, s, \tau) = \sin s \sin \tau. \tag{10}$$

Discuss the behavior of the solution. (Hint: the solution is given by $u = \sin(y - x\, u) \sin(z - x\, u)$).

Problem 5. (Steepening of Waves) Consider the Burgers equation

$$u_t + u\, u_x = 0, \qquad u(0, x) = u_0(x). \tag{11}$$

In the lecture we have shown that the characteristics, which are straight lines, will cross each other, causing problem in defining a single valued solution. However, if we drop the "single value" requirement, the solution, as a multi-valued function, can exists up to any late time (there are situations where multi-valuedness is not a problem: Think of those waves surfers ride on).

Now assume $u_0(x) \in C_0^\infty$ (smooth with compact support – vanish outside a finite interval). Then the area enclosed by the graph of u and the x-axis is well-defined. Show that this area is conserved by the above multi-valued solution.

(Hint: A simple proof I found is to show that the boundary of this region moves by a divergence-free velocity field, and use incompressibility of such 2D flows.)

Problem*.