Math 527 Lecture 2: Conservation Laws I

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Conservation laws.

- Evolution of the density ρ of a certain substance:
  - The total amount inside a set Ω at time t:
$$\int_\Omega \rho(t,x)\,\mathrm{d}x. \tag{1}$$
  - Assumption: Change only occurs as this substance goes through the boundary. Quantified by a "flux" F. That is
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(t,x)\,\mathrm{d}x = -\int_{\partial\Omega} F\cdot n\,\mathrm{d}s \tag{2}$$
where n is the outer normal.
  - Gauss Theorem:
$$\int_{\partial\Omega} F\cdot n\,\mathrm{d}s = \int_\Omega \nabla\cdot F\,\mathrm{d}x \tag{3}$$
  - We have
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(t,x)\,\mathrm{d}x + \int_\Omega \nabla\cdot F\,\mathrm{d}x = 0 \tag{4}$$
as $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(t,x) = \int_\Omega \rho_t\,\mathrm{d}x$, we reach
$$\int_\Omega [\rho_t + \nabla\cdot F]\,\mathrm{d}x = 0. \tag{5}$$
  - As Ω is arbitrary, when $\rho_t + \nabla\cdot F$ is assumed to be regular enough, we have a differential equation
$$\rho_t + \nabla\cdot F = 0. \tag{6}$$
- Conservation Laws.
  - A conservation law is obtained when F is a function of ρ (from now on we use u instead):
$$u_t + \nabla\cdot F(u) = 0, \qquad u(0,x) = g(x) \tag{7}$$
  - In particular, a 1D scalar conservation law is
$$u_t + f(u)_x = 0, \qquad u(0,x) = g(x) \tag{8}$$
- Solving conservation laws using method of characteristics.
  - Denote $a(u) = f'(u)$. We can write the equation as
$$u_t + a(u)\,u_x = 0. \tag{9}$$
the method of characteristics gives
$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1, \qquad t(0) = 0 \tag{10}$$
$$\frac{\mathrm{d}x}{\mathrm{d}s} = a(u), \qquad x(0) = x_0 \tag{11}$$
$$\frac{\mathrm{d}u}{\mathrm{d}s} = 0, \qquad u(0) = g(x_0) \tag{12}$$
The solution is given by
$$u(t,x) = g(x_0) \tag{13}$$

with

$$x = x_0 + a(g(x_0))\, t \tag{14}$$

Combine these we get an implicit formula for u:

$$u = g(x - a(u)\, t). \tag{15}$$

- Although under certain conditions the implicit function theorem gives the existence of a uniquely determined u, we cannot really write it down explicitly.
- However we can compute u_x explicitly:

$$u_x = g'\,(1 - a'\, t\, u_x) \Longrightarrow u_x = \frac{g'}{1 + g'\, a'\, t}. \tag{16}$$

We see that there is the possibility that u_x will become infinity – when $g'\, a' < 0$.

- Indeed, consider the Burgers equation

$$u_t + u\, u_x = 0 \tag{17}$$

with

$$u(0, x) = g(x) = \begin{cases} 0 & x < -1 \\ x + 1 & -1 < x < 0 \\ 1 - x & 0 < x < 1 \\ 0 & x > 1 \end{cases} \tag{18}$$

we see that characteristics will cross each other, leading to multi-valued solutions which are usually non-physical.

- Weak solution.
  - We would like to define "weak solutions" for the equation[1]

$$u_t + f(u)_x = 0, \qquad u(x, 0) = u_0(x). \tag{19}$$

As any meaningful definition of "weak solutions" should coincide with the classical definition when the solution is smooth enough, we multiply the equation by a C^1 test function ϕ and integrate by parts as if u is C^1.

$$[u_t + f(u)_x]\, \phi(x, t) = 0 \implies -\int\!\!\int_\Omega u\, \phi_t + f(u)\, \phi_x \,\mathrm{d}x\, \mathrm{d}t + \oint_{\partial\Omega} \phi(x, t)\, [u\, n_t + f(u)\, n_x]\, \mathrm{d}S = 0. \tag{20}$$

If we take Ω to be the intersection of the support of ϕ and the half-plane $t > 0$, we obtain

$$\int\!\!\int_{t>0} u\, \phi_t + f(u)\, \phi_x \,\mathrm{d}x\, \mathrm{d}t + \int_{\mathbb{R}} u_0\, \phi \,\mathrm{d}x = 0. \tag{21}$$

Notice that u no longer needs to be C^1 to make the above integrals meaningful. For $\phi \in C^1$, the only requirement we should put on u is that both u, $f(u)$ are measures. In particular, it is OK for u to be piecewise continuous.

Definition 1. *u is called a weak solution of*

$$u_t + f(u)_x = 0, \qquad u(x, 0) = u_0 \tag{22}$$

if

$$\int\!\!\int_{t>0} u\, \phi_t + f(u)\, \phi_x \,\mathrm{d}x\, \mathrm{d}t + \int_{\mathbb{R}} u_0\, \phi \,\mathrm{d}x = 0. \tag{23}$$

holds for any $\phi \in C_0^1$.

1. For simplicity we deal with the scalar case here. We should keep in mind that the "real-world" problems are mostly systems of conservation laws. For them weak solutions can be defined similarly.

One can easily show that if $u \in C^1$ is a weak solution, then it also solves the equation in the classical sense.

- Note that, the above definition allows discontinuity but not multi-valuedness.
- The goal of introducing weak solution is to do more than method of characteristics, which are faced with two problems:
  1. The method of characteristics may leave regions of solution undefined. Consider the example:
  $$u_t + u\, u_x = 0, \qquad u(0, x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}. \tag{24}$$
  2. As already illustrated, the method of characteristics may cause ambiguity in the value of u.

- Filling the void: Rarefaction waves.
  - We claim that
  $$u(t, x) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < t \\ 1 & x > t \end{cases} \tag{25}$$
  is a weak solution to the problem
  $$u_t + u\, u_x = 0, \qquad u(0, x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}. \tag{26}$$
  - More examples in next lecture.
- Resolving the ambiguity: Shock waves.
  - Instead of letting characteristics cross, we introduce a discontinuity curve which "cuts" the characteristics. That is, we try to get u which is piecewise C^1 with jump discontinuities along certain curves. Turns out that the requirement of u being a weak solution totally determines these curves.
  - Consider one such curve, denote it by Γ. Let $\phi \in C_0^1$ be supported in a small ball centering on Γ. The ball is so small that it does not intersect with the x-axis and u is C^1 everywhere in the ball except along Γ.

    Denote this ball by D, which is divided into two parts D_1, D_2 by Γ. As $\phi = 0$ along the x-axis, the definition of weak solutions becomes
    $$\int\int_D u\, \phi_t + f(u)\, \phi_x \,\mathrm{d}x\, \mathrm{d}t = 0. \tag{27}$$
    We write the left hand side as $\int \int_{D_1} + \int \int_{D_2}$ and try to use integration by parts.

    Since u is C^1 in D_1, D_2, we have
    $$\int\int_{D_1} u\, \phi_t + f(u)\, \phi_x \,\mathrm{d}x\, \mathrm{d}t = -\int\int [u_t + f(u)_x]\, \phi \,\mathrm{d}x\, \mathrm{d}t + \oint_{\partial D_1} [u\, n_t + f(u)\, n_x]\, \phi \,\mathrm{d}S \tag{28}$$
    Since u solves the equation in the classical sense in D_1 (see exercise) we have
    $$\int\int_{D_1} u\, \phi_t + f(u)\, \phi_x \,\mathrm{d}x\, \mathrm{d}t = \oint_{\partial D_1} [u\, n_t + f(u)\, n_x]\, \phi \,\mathrm{d}S. \tag{29}$$
    Similarly
    $$\int\int_{D_2} u\, \phi_t + f(u)\, \phi_x \,\mathrm{d}x\, \mathrm{d}t = \oint_{\partial D_2} [u\, n_t + f(u)\, n_x]\, \phi \,\mathrm{d}S. \tag{30}$$
    Since ϕ vanishes on ∂D_1 except along Γ, we finally obtain
    $$\int_\Gamma [[u]\, n_t + [f(u)]\, n_x]\, \phi \,\mathrm{d}S = 0 \tag{31}$$

where $[u]$ is the "jump" of u across Γ.

Now let Γ be determined by $\frac{\mathrm{d}x}{\mathrm{d}t} = s(x,t)$. We have $\frac{n_t}{n_x} = -s$ which gives

$$\int_\Gamma [-s\,[u] + [f(u)]]\,\phi\,\mathrm{d}S = 0. \tag{32}$$

Due to the arbitrariness of ϕ, the weak solution must satisfy

$$[f(u)] = s\,[u]. \tag{33}$$

This is called the *jump condition*. On can also do the same analysis for systems of conservation laws and obtain

$$[\boldsymbol{f}(\boldsymbol{u})] = s\,[\boldsymbol{u}]. \tag{34}$$

In the special case of gas dynamics, this condition is referred to as *Rankine-Hugoniot condition*.