Conservation laws.

- Evolution of the density \( \rho \) of a certain substance:
  - The total amount inside a set \( \Omega \) at time \( t \):
    \[
    \int_{\Omega} \rho(t, x) \, dx.
    \]  
  - Assumption: Change only occurs as this substance goes through the boundary. Quantified by a "flux" \( F \). That is
    \[
    \frac{d}{dt} \int_{\Omega} \rho(t, x) \, dx = -\int_{\partial\Omega} F \cdot n \, ds
    \]  
    where \( n \) is the outer normal.
- Gauss Theorem:
  \[
  \int_{\partial\Omega} F \cdot n \, ds = \int_{\Omega} \nabla \cdot F \, dx
  \]  
- We have
  \[
  \frac{d}{dt} \int_{\Omega} \rho(t, x) \, dx + \int_{\Omega} \nabla \cdot F \, dx = 0
  \]  
  as \( \frac{d}{dt} \int_{\Omega} \rho(t, x) \, dx = \int_{\Omega} \rho_t \, dx \), we reach
  \[
  \int_{\Omega} [\rho_t + \nabla \cdot F] \, dx = 0.
  \]
- As \( \Omega \) is arbitrary, when \( \rho_t + \nabla \cdot F \) is assumed to be regular enough, we have a differential equation
  \[
  \rho_t + \nabla \cdot F = 0.
  \]

- Conservation Laws.
  - A conservation law is obtained when \( F \) is a function of \( \rho \) (from now on we use \( u \) instead):
    \[
    u_t + \nabla \cdot F(u) = 0, \quad u(0, x) = g(x)
    \]  
  - In particular, a 1D scalar conservation law is
    \[
    u_t + f(u)_x = 0, \quad u(0, x) = g(x)
    \]

- Solving conservation laws using method of characteristics.
  - Denote \( a(u) = f'(u) \). We can write the equation as
    \[
    u_t + a(u) \ u_x = 0.
    \]  
  the method of characteristics gives
    \[
    \frac{dt}{ds} = 1, \quad t(0) = 0
    \]
    \[
    \frac{dx}{ds} = a(u), \quad x(0) = x_0
    \]
    \[
    \frac{du}{ds} = 0, \quad u(0) = g(x_0)
    \]
  The solution is given by
  \[
  u(t, x) = g(x_0)
  \]
with
\[ x = x_0 + a(g(x_0)) t \]  
(14)

Combine these we get an implicit formula for \( u \):
\[ u = g(x - a(u) t). \]  
(15)

- Although under certain conditions the implicit function theorem gives the existence of a uniquely determined \( u \), we cannot really write it down explicitly.
- However we can compute \( u_x \) explicitly:
\[ u_x = g'(1 - a' t u_x) / u_x = g' / (1 + g' a'), \]  
(16)

We see that there is the possibility that \( u_x \) will become infinity – when \( g' a' < 0 \).
- Indeed, consider the Burgers equation
\[ u_t + u u_x = 0 \]  
(17)

with
\[ u(0, x) = g(x) = \begin{cases} 
0 & x < -1 \\
x + 1 & -1 < x < 0 \\
1 - x & 0 < x < 1 \\
0 & x > 1 
\end{cases} \]  
(18)

we see that characteristics will cross each other, leading to multi-valued solutions which are usually non-physical.

- **Weak solution.**
  - We would like to define “weak solutions” for the equation\(^1\)
\[ u_t + f(u)_x = 0, \quad u(x, 0) = u_0(x). \]  
(19)

As any meaningful definition of “weak solutions” should coincide with the classical definition when the solution is smooth enough, we multiply the equation by a \( C^1 \) test function \( \phi \) and integrate by parts as if \( u \) is \( C^1 \).
\[ [u_t + f(u)_x] \phi(x, t) = 0 \implies - \int \int u \phi_t + f(u) \phi_x \, dx \, dt + \int_\partial \phi(x, t) [u n_t + f(u) n_x] \, dS = 0. \]  
(20)

If we take \( \Omega \) to be the intersection of the support of \( \phi \) and the half-plane \( t > 0 \), we obtain
\[ \int \int_{t>0} u \phi_t + f(u) \phi_x \, dx \, dt + \int_{\partial \Omega} u_0 \phi \, dx = 0. \]  
(21)

Notice that \( u \) no longer needs to be \( C^1 \) to make the above integrals meaningful. For \( \phi \in C^1 \), the only requirement we should put on \( u \) is that both \( u, f(u) \) are measures. In particular, it is OK for \( u \) to be piecewise continuous.

**Definition 1.** \( u \) is called a weak solution of
\[ u_t + f(u)_x = 0, \quad u(x, 0) = u_0 \]  
(22)

if
\[ \int \int_{t>0} u \phi_t + f(u) \phi_x \, dx \, dt + \int_{\mathbb{R}} u_0 \phi \, dx = 0. \]  
(23)

holds for any \( \phi \in C^1_0 \).

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1. For simplicity we deal with the scalar case here. We should keep in mind that the “real-world” problems are mostly systems of conservation laws. For them weak solutions can be defined similarly.
One can easily show that if \( u \in C^1 \) is a weak solution, then it also solves the equation in the classical sense.

- Note that, the above definition allows discontinuity but not multi-valuedness.
- The goal of introducing weak solution is to do more than method of characteristics, which are faced with two problems:
  1. The method of characteristics may leave regions of solution undefined. Consider the example:
     \[
     u_t + u u_x = 0, \quad u(0, x) = \begin{cases} 
     0 & x < 0 \\
     1 & x > 0 
     \end{cases}.
     \] (24)
  2. As already illustrated, the method of characteristics may cause ambiguity in the value of \( u \).

- Filling the void: Rarefaction waves.
  - We claim that
    \[
    u(t, x) = \begin{cases} 
    0 & x < 0 \\
    x/t & 0 < x < t \\
    1 & x > t 
    \end{cases}
    \] (25)
    is a weak solution to the problem
    \[
    u_t + u u_x = 0, \quad u(0, x) = \begin{cases} 
    0 & x < 0 \\
    1 & x > 0 
    \end{cases}.
    \] (26)
  - More examples in next lecture.

- Resolving the ambiguity: Shock waves.
  - Instead of letting characteristics cross, we introduce a discontinuity curve which “cuts” the characteristics. That is, we try to get \( u \) which is piecewise \( C^1 \) with jump discontinuities along certain curves. Turns out that the requirement of \( u \) being a weak solution totally determines these curves.
  - Consider one such curve, denote it by \( \Gamma \). Let \( \phi \in C_0^1 \) be supported in a small ball centering on \( \Gamma \). The ball is so small that it does not intersect with the \( x \)-axis and \( u \) is \( C^1 \) everywhere in the ball except along \( \Gamma \).
    Denote this ball by \( D \), which is divided into two parts \( D_1, D_2 \) by \( \Gamma \). As \( \phi = 0 \) along the \( x \)-axis, the definition of weak solutions becomes
    \[
    \int \int_D u \phi_t + f(u) \phi_x \, dx \, dt = 0.
    \] (27)
    We write the left hand side as \( \int \int_{D_1} + \int \int_{D_2} \) and try to use integration by parts.
    Since \( u \) is \( C^1 \) in \( D_1, D_2 \), we have
    \[
    \int \int_{D_1} u \phi_t + f(u) \phi_x \, dx \, dt = - \int \int_{D_1} [u_t + f(u) u_x] \phi \, dx \, dt + \int \int_{\partial D_1} [u n_t + f(u) n_x] \phi \, dS
    \] (28)
    Since \( u \) solves the equation in the classical sense in \( D_1 \) (see exercise) we have
    \[
    \int \int_{D_1} u \phi_t + f(u) \phi_x \, dx \, dt = \int \int_{\partial D_1} [u n_t + f(u) n_x] \phi \, dS.
    \] (29)
    Similarly
    \[
    \int \int_{D_2} u \phi_t + f(u) \phi_x \, dx \, dt = \int \int_{\partial D_2} [u n_t + f(u) n_x] \phi \, dS.
    \] (30)
    Since \( \phi \) vanishes on \( \partial D_1 \) except along \( \Gamma \), we finally obtain
    \[
    \int_{\Gamma} [u n_t + f(u) n_x] \phi \, dS = 0
    \] (31)
where $[u]$ is the “jump” of $u$ across $\Gamma$.

Now let $\Gamma$ be determined by $\frac{dx}{dt} = s(x,t)$. We have $\frac{\partial u}{\partial x} = -s$ which gives

$$\int_{\Gamma} [-s[u] + [f(u)]] \phi \, dS = 0.$$  \hspace{1cm} (32)

Due to the arbitrariness of $\phi$, the weak solution must satisfy

$$[f(u)] = s[u].$$  \hspace{1cm} (33)

This is called the jump condition. On can also do the same analysis for systems of conservation laws and obtain

$$[f(u)] = s[u].$$  \hspace{1cm} (34)

In the special case of gas dynamics, this condition is referred to as Rankine-Hugoniot condition.