Math 527 Lecture 1 Exercises and Problems

Exercises.

Exercise 1. Find a relation between the Monge-Ampere equation

$$\det(D^2 u) = f \tag{1}$$

and the Poisson equation

$$\triangle u = f. \tag{2}$$

Exercise 2. Consider the eikonal equation

$$|Du| = 1 \qquad x \in \Omega; \qquad u = 0 \qquad x \in \partial\Omega. \tag{3}$$

Here $|Du| := \sqrt{u_{x_1}^2 + \cdots + u_{x_n}^2}$. Show that there can be no solution $u \in C^1$ (having continuous derivatives). (Hint: Consider the maximum and minimum of u).

Problems.

Problem 1. State and prove the well-posedness theory for the general ODE system

$$\dot{x} = b(t, x) \tag{4}$$

Here x and b are both vector valued.

Problem 2. (Evans PDE) Solve the following equations using method of characteristics:

a) $x_1 u_{x_1} + x_2 u_{x_2} = 2u, \qquad u(x_1, 1) = g(x_1).$

b) $x_1 u_{x_1} + 2 x_2 u_{x_2} + u_{x_3} = 3u, \qquad u(x_1, x_2, 0) = g(x_1, x_2).$

Problem 3. Consider the following equation:

$$a(t, x)\, u_t + b(t, x) \cdot \nabla u = f(t, x, u). \tag{5}$$

Solve it using the method of characteristics. What conditions should be put on a, b, f? Why? (You don't need to rigorously prove your claims).

Problem 4. (Evans PDE) Given a smooth vector field b on $\mathbb{R}^n$. Let $X(s) = X(s, x, t)$ solve the ODE

$$\dot{X} = b(X), \qquad X(t) = x \tag{6}$$

a) Define the Jacobian

$$J(s, x, t) = \det D_x X(s, x, t) \tag{7}$$

and derive Euler's formula

$$J_s = (\nabla \cdot b)(X)\, J. \tag{8}$$

b) Demonstrate that

$$u(x, t) := g(X(0, x, t))\, J(0, x, t) \tag{9}$$

solves

$$u_t + \nabla \cdot (b\, u) = 0, \qquad u(0, x) = g \tag{10}$$

(Hint: Show $\frac{\partial}{\partial s}(u(X, s)\, J) = 0$)

Problem 5. (Thanks to the question from Mr. M. Slevinsky) Solve the following transport equation:

$$u_t + (\sin x)\, u_x = 0, \qquad u(0, x) = \delta(x) \tag{11}$$

Here $\delta(x)$ is the Dirac delta function.

Problem*.

Problem 6. Consider the 2D Laplace equation with "initial value":

$$u_{xx} + u_{yy} = 0, \qquad u(x, 0) = u_0(x), \qquad u(0, y) = u(2\pi, y) = 0. \tag{12}$$

Study in detail its well-posedness. In particular, is the solution unique?