# Math 527 Lecture 1: Introduction

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**Introduction.**

- There is no general theory for PDE. Therefore have to study the important ones only. Fortunately, understanding of many important PDEs can be gained through thorough understanding of several "paradigm equations".
- Paradigm equations:
  - Transport equation.
  $$u_t + b \cdot \nabla u = 0 \tag{1}$$
  Here there are several cases:
    - Linear: $b = b(t, x)$;
    - Quasilinear: $b = b(t, x, u)$;
    - Nonlinear: $b = b(t, x, u, \nabla u)$.

    A "rule of thumb" of PDE is that the most important term in a PDE is the highest order term. Therefore when $b$ depends on higher order derivatives, writing the equation in the above form is not meaningful anymore.
  - Laplace/Poisson equation.
  $$\triangle u = u_{x_1 x_1} + \cdots + u_{x_n x_n} = 0 \text{ (Laplace) or } f \text{ (Poisson)} \tag{2}$$
  - Heat equation:
  $$u_t = k \triangle u \tag{3}$$
  - Wave equation:
  $$u_{tt} = k \triangle u. \tag{4}$$

  We will try to gain understanding of the above four paradigm equations.
- Why are paradigm equations important?
  - They are "building blocks" to more complicated equations.
  - Black-Scholes equation from Mathematical Finance:
  $$u_t + \frac{1}{2} \sigma^2 x^2 u_{xx} + r x u_x - r u = 0 \tag{5}$$
  Decompose:
    - $u_t + \frac{1}{2} \sigma^2 x^2 u_{xx} = \cdots$: Relates to heat equation[1];
    - $u_t + r x u_x = \cdots$: Relates to transport equation;
    - $u_t - r u = \cdots$: Relates to ODE.
  - Minimal surface equation:
  $$\nabla \cdot \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \tag{6}$$
  Compare with the Laplace equation:
  $$\nabla \cdot \left( \frac{\nabla u}{1} \right) = 0. \tag{7}$$

1. With negative diffusion!

- The very first thing to study: Well-posedness.
  - Existence: There is/are solution(s).
  - Uniqueness: There is only one solution.
  - Continuous dependence on data. This means, if we change the initial value or boundary value a little bit, the solution also changes just a little bit.
- Classical example of ill-posedness: Initial value problem for Laplace equation. What fails is continuous dependence on data.[2]

  The ill-posedness stems from "wrong" type of data. If instead we specify boundary value, that is

$$\triangle u = 0 \qquad \text{in } \Omega; \qquad u = g \qquad \text{on } \partial\Omega \tag{8}$$

  then the problem is perfectly well-posed.
- Example of PDE without solutions no matter what boundary values are specified:

$$u_x + i\, u_y - 2\, i\, (x + i\, y)\, u_t = f(x, y, t). \tag{9}$$

  there are $f(x, y, t) \in C^\infty$ such that no $C^1$ solution exists anywhere.

**Linear Transport Equations.**

$$u_t + b(t, x) \cdot \nabla u = 0, \qquad u(0, x) = u_0(x). \tag{10}$$

- Method of characteristics.

  Idea: Find special curve $(t(s), x(s))$ along which $u_t + b(t, x) \cdot \nabla u = \frac{\mathrm{d}u}{\mathrm{d}s}$.
  - Chain rule:

$$\frac{\mathrm{d}u(t(s), x(s))}{\mathrm{d}s} = \frac{\mathrm{d}t}{\mathrm{d}s}\, u_t + \frac{\mathrm{d}x}{\mathrm{d}s} \cdot \nabla u \tag{11}$$

    Compare with the equation:

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1, \qquad \frac{\mathrm{d}x}{\mathrm{d}s} = b(t, x). \tag{12}$$

  - Note that the above system can be solved, given that $b$ is Lipschitz in $x$.
  - Let $x(s), t(s)$ be the solution, then we have

$$u(t(s), x(s)) = u_0(x_0). \tag{13}$$

  - In the last step, try to invert the relation

$$\begin{pmatrix} s \\ x_0 \end{pmatrix} \longrightarrow \begin{pmatrix} t \\ x \end{pmatrix} \tag{14}$$

    to get

$$x_0 = x_0(t, x). \tag{15}$$

    Then the solution is given by

$$u(t, x) = u_0(x_0(t, x)). \tag{16}$$

  **Example 1.** $b$ is constant. The characteristic system is

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1 \Longrightarrow t = s; \qquad \frac{\mathrm{d}x}{\mathrm{d}s} = b \Longrightarrow x = x_0 + b\, s. \tag{17}$$

  The invertion is easy:

$$x_0 = x - b\, s = x - b\, t. \tag{18}$$

  So the solution is given by

$$u(t, x) = u_0(x - b\, t). \tag{19}$$

2. See the *ed problem for this lecture!

Intuition: The initial value $u_0$ is "transported" in the $x$-space with velocity $b$.

- Generalizations.
  - Irregular $u_0$. There is no problem carrying the above out when $u_0$ is not differentiable, as long as the composition $u_0(x_0(t,x))$ makes sense. Therefore, the whole approach should still work for initial values that are measures. But may break down for $u_0$ just a distribution.[3]
  - Irregular $b$. This is much more complicated. The celebrated Lions-DiPerna theory establishes well-posedness for $b \in L^\infty$ with $\nabla \cdot b \in L^\infty$ and $\nabla b \in L^1$. More recently Ambrosio weakened the conditions to $b \in \mathrm{BV}$.

    The reason for studying irregular $b$ is the hope of applying linear theory to quasi-linear cases ($b = b(t,x,u)$). In that case we may not know a priori that $b$ is Lipschitz.

3. Guess: $b \in C^{k+1}$ then $u_0$ can be a distribution of order $k$.