4. Special Solutions and Stability

Most PDEs in science and engineering are nonlinear. They cannot be solved either explicitly or semi-explicitly (separation of variables, transform methods, Green's functions, etc.). For these equations one important strategy is to explicitly find special solutions, and then try to study the behavior of solutions close to these solutions through "linearization" – Replacing the equation by a linear equation which approximates the original equation when u is close enough to the special solution.

4.0. Approximate solutions and stability for ODEs.

We start from the initial value problem of first order ODE system:

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{4.1}$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{x}_0 = (x_{01}, \ldots, x_{0n})$. Recall that the initial value problem of a single ordinary differential equations of order n

$$F\left(x, \dot{x}, \ddot{x}, \ldots, \frac{\mathrm{d}^n x}{\mathrm{d}t^n}, t\right) = 0, \qquad x(0) = x_{01},\ \dot{x}(0) = x_{02}, \ldots \tag{4.2}$$

can be reduced to a first order system through setting

$$x_1 = x, \qquad x_2 = \dot{x}, \qquad x_3 = \ddot{x}, \ldots \tag{4.3}$$

If the function $\boldsymbol{F}(\boldsymbol{x}, t)$ does not explicitly involve the variable t, we say the system is "autonomous". For example, the Lorenz system (the simplest example for chaotic behavior)

$$\dot{x} = 10(y - x), \qquad \dot{y} = x\,(28 - z) - y, \qquad \dot{z} = x\,y - \frac{8}{3}z \tag{4.4}$$

is an autonomous system with $\boldsymbol{x} = (x, y, z)$ and

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{pmatrix} 10(y - x) \\ x\,(28 - z) - y \\ x\,y - \frac{8}{3}z \end{pmatrix}. \tag{4.5}$$

Autonomous systems are easier to analyze than non-autonomous ones, although the latter can always to transformed to the former – at the cost of linear systems may be transformed to nonlinear ones.

The study of an autonomous nonlinear systems often proceed as follows:

1. Find the stationary solutions – solutions that are independent of the variable t;
2. Study the stability around these solutions to deduce local behaviors of solutions;
3. Try to conclude global behavior based on local ones.

It should be noted that the above analysis is usually done in the "phase space", that is the $\boldsymbol{x}$ space, a solution is then represented as a curve in the phase space. This would not be possible if the ODE system is not-autonomous.

4.0.1. Stationary solutions.

A stationary solution is a solution that is a constant, that is $\dot{\boldsymbol{x}} = 0$. Recalling the system $\dot{\boldsymbol{x}} = F(\boldsymbol{x})$, we see that any stationary solution must be a root to the equation

$$F(\boldsymbol{x}) = 0 \tag{4.6}$$

and the converse is also true: If a constant vector solves $\boldsymbol{F}(\boldsymbol{c}) = 0$, then $\boldsymbol{x}(t) = \boldsymbol{c}$ is a solution to the system.

Example 4.1. For the Lorenz system,

$$\dot{x} = 10(y - x), \qquad \dot{y} = x\,(28 - z) - y, \qquad \dot{z} = x\,y - \frac{8}{3}z \tag{4.7}$$

we can find all stationary solutions by solving

$$\begin{aligned} 10\,(y-x) &= 0 && (4.8)\\ x\,(28-z)-y &= 0 && (4.9)\\ x\,y-\frac{8}{3}z &= 0. && (4.10) \end{aligned}$$

This gives

$$x=y=\pm 6\sqrt{2}, \qquad z=27. \tag{4.11}$$

Thus there are two stationary solutions:

$$(x,y,z)=\left(6\sqrt{2},6\sqrt{2},27\right); \qquad (x,y,z)=\left(-6\sqrt{2},-6\sqrt{2},27\right). \tag{4.12}$$

The integral curves of these two solutions in the x-y-z space are simply the two dots at the above locations.

4.0.2. Linearization and stability analysis.

The key fact in the study of linear ODE systems is that, if the system is linear with constant-coefficients, then we can solve it explicitly (On the other hand, once the coefficients involve the variable, in general no simple formula is available).

Theorem 4.2. *Consider a linear ODE system with constant-coefficients,*

$$\begin{aligned} \dot{x}_1 &= a_{11}\,x_1+\cdots+a_{1n}\,x_n && (4.13)\\ \dot{x}_2 &= a_{21}\,x_1+\cdots+a_{2n}\,x_n && (4.14)\\ \vdots &\ \ \vdots\ \ \vdots \\ \dot{x}_n &= a_{n1}\,x_1+\cdots+a_{nn}\,x_n. && (4.15) \end{aligned}$$

Or in matrix form

$$\dot{\boldsymbol{x}}=A\,\boldsymbol{x}, \qquad A=\begin{pmatrix} a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nn} \end{pmatrix}. \tag{4.16}$$

Then unique solution is given by

$$\boldsymbol{x}(t)=e^{A\,t}\,\boldsymbol{x}_0 \tag{4.17}$$

where $\boldsymbol{x}_0$ is the initial condition.

Recall that the matrix exponential is defined as follows:

$$e^B=I+B+\frac{B^2}{2}+\cdots+\frac{B^n}{n!}+\cdots=\sum_{n=0}^{\infty}\frac{B^n}{n!}. \tag{4.18}$$

From this we easily see that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[e^{A\,t}\,\boldsymbol{x}_0] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{n=0}^{\infty}\frac{A^n\,t^n}{n!}\right]\boldsymbol{x}_0\\ &= \sum_{n=1}^{\infty}\frac{A^n\,t^{n-1}}{(n-1)!}\boldsymbol{x}_0\\ &= A\left[\sum_{n=0}^{\infty}\frac{(A\,t)^n}{n!}\boldsymbol{x}_0\right]=A\,[e^{A\,t}\,\boldsymbol{x}_0]. && (4.19) \end{aligned}$$

and

$$[e^{A\,t}\,\boldsymbol{x}_0]_{t=0}=\sum_{n=0}^{\infty}\frac{A^n\,0^n}{n!}\boldsymbol{x}_0=I\,\boldsymbol{x}_0=\boldsymbol{x}_0. \tag{4.20}$$

The advantage of writing the solution in matrix form is the following:

Lemma 4.3. *If* $A = P^{-1} B P$ *where* P *is an* $n \times n$ *nonsingular matrix, then*

$$e^{At} = P^{-1} e^{Bt} P. \tag{4.21}$$

Proof. We have, by definition,

$$\begin{aligned} e^{At} &= \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(P^{-1} B P)^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{P^{-1} B^n P t^n}{n!} \\ &= \sum_{n=0}^{\infty} P^{-1} \left(\frac{B^n t^n}{n!} \right) P \\ &= P^{-1} \left[\sum_{n=0}^{\infty} \frac{B^n t^n}{n!} \right] P \\ &= P^{-1} e^{Bt} P. \end{aligned} \tag{4.22}$$

Thus ends the proof. □

Thus for example, if the matrix A is diagonalizable, that is we can find P such that $A = P^{-1} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P$, then

$$e^{At} = P^{-1} \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} P \tag{4.23}$$

and the solution now becomes

$$\boldsymbol{x}(t) = P^{-1} \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} P \boldsymbol{x}_0 \tag{4.24}$$

whose behavior is particular easy to understand if we introduce a new coordinate system

$$\boldsymbol{y} = P \boldsymbol{x}. \tag{4.25}$$

In this new coordinate system we have

$$\boldsymbol{y}(t) = \begin{pmatrix} e^{\lambda_1 t} y_{01} \\ \vdots \\ e^{\lambda_n t} y_{0n} \end{pmatrix}. \tag{4.26}$$

We see that the behavior of $\boldsymbol{y}$ depends on the signs of the real parts of the eigenvalues λ_1, ..., λ_n. More specifically, if only one of the initial values y_{0i} is nonzero, then $|\boldsymbol{y}|$ increases when $\Re \lambda_i > 0$ and decreases when $\Re \lambda_i < 0$, it stays the same when $\Re \lambda_i = 0$.

For a general $n \times n$ matrix A, the best we can do is to reduce it to Jordan canonical form:

$$A = P^{-1} \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix} P \tag{4.27}$$

where each "Jordan block" is of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix} \tag{4.28}$$

where all the empty entries are 0's. It turns out that we can explicitly calculate

$$e^{J_i t} = e^{\lambda_i t} \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^m}{m!} \\ & 1 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2} \\ & & & \ddots & t \\ & & & & 1 \end{pmatrix} \tag{4.29}$$

where m is the size of the Jordan block.

Thus the solution to

$$\dot{\boldsymbol{x}} = A\,\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{4.30}$$

reads

$$\boldsymbol{x}(t) = P^{-1} \begin{pmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_k t} \end{pmatrix} P \boldsymbol{x}_0. \tag{4.31}$$

If we introduce $\boldsymbol{y} = P\boldsymbol{x}$ we have

$$\boldsymbol{y}(t) = \begin{pmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_k t} \end{pmatrix} \boldsymbol{y}_0. \tag{4.32}$$

We see that we still have

- If all $\Re\lambda_i < 0$, then the all solutions are "attracted" to the origin;
- If one $\Re\lambda_i > 0$, then some (in fact, most) solutions are "moving away" from the origin and towards infinity.

Therefore we have the following important observation: The stability of a linear system of ODEs with constant coefficients

$$\dot{\boldsymbol{x}} = A\,\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{4.33}$$

is determined by the signs of the real parts of the eigenvalues of A.

Remark 4.4. Note that for a linear system, the only stationary solution is the origin.

How can we apply this understanding of the linear system to nonlinear systems? The basic idea is linearization, that is, around each stationary solution we find a linear system that approximates the nonlinear system there. Naturally, we linearize as follows.

Linearization. Let $\boldsymbol{x}(t) = \boldsymbol{x}_0$ be a stationary solution for an autonomous system

$$\dot{\boldsymbol{x}} = F(\boldsymbol{x}). \tag{4.34}$$

Then the linearization at $\boldsymbol{x}_0$ is given by

$$\dot{\boldsymbol{y}} = A\,\boldsymbol{y} \tag{4.35}$$

where $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{x}_0$ and

$$A = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix} \tag{4.36}$$

where the matrix is evaluated at $\boldsymbol{x}_0$, so A is a constant matrix.

Once this is done, we can study the eigenvalues of the matrix A and obtain the local behavior of the solutions. In particular, if all the eigenvalues has negative real parts, we say the system is **linearly stable at $\boldsymbol{x}_0$**; If at least one eigenvalue has positive real part, we say the system is **linearly unstable at $\boldsymbol{x}_0$**.

Example 4.5. Study

$$\dot{x} = x - y, \qquad \dot{y} = 1 - x\,y. \tag{4.37}$$

Solution. First we find stationary solutions:

$$x-y=0, \qquad 1-x\,y=0 \Longrightarrow (1,1),(-1,-1). \tag{4.38}$$

- Linearize at $(1,1)$: We can easily calculate the Jacobian matrix to be

$$A=\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}. \tag{4.39}$$

Let $\xi=x-1, \eta=y-1$ we obtain the local linearized system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \xi \\ \eta \end{pmatrix}=\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{4.40}$$

Solving

$$\det(\lambda\,I-A)=0 \tag{4.41}$$

gives the eigenvalues as $\pm\sqrt{2}$ which gives a saddle.

- Linearize at $(-1,-1)$. This time

$$A=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{4.42}$$

and the eigenvalues are $1\pm i$. Thus the stationary solution $(1,1)$ is unstable.

4.0.3. From local to global.

This is tricky but sometimes can be done. One key fact is that integral curves cannot meet one another unless at a point representing a stationary solution (or at infinity).

4.0.4. PDEs?.

For PDEs, there are many more types of special solutions, with stationary solutions the simplest type. In particular, one can look for "self-similar" solutions such as $u(x,y)=\Phi(\xi)$ with $\xi=x^a\,y^b$, or "travelling wave" solutions such as $u(x,t)=\Phi(x-c\,t)$.

In principle, we can carry out the above around all types of special solutions. However the linearized equation is often too hard to study when linearizing around non-stationary solutions. Therefore in the following we will only focus on linearization around stationary solutions.

References.

- Terence Tao, "Nonlinear Dispersive Equations: Local and Global Analysis", §1.1.

Exercises.

Exercise 4.1. Consider a non-autonomous system $\dot{\boldsymbol{x}}=\boldsymbol{F}(\boldsymbol{x},t)$. Introduce a new variable $x_{n+1}=t$ to make it autonomous. Show with an example that a linear system may be transformed to a nonlinear one through the above process.

Exercise 4.2. Consider an autonomous system in $\mathbb{R}^n$, that is $\boldsymbol{x}=(x_1,\ldots,x_n)$,

$$\dot{\boldsymbol{x}}=F(\boldsymbol{x}). \tag{4.43}$$

Prove that if two integral curves touch, the intersection point must be a stationary solution.

Exercise 4.3. (Properties of matrix exponentials) Let A,B be $n\times n$ matrices.

a) Prove that if $A\,B=B\,A$, then

$$e^A\,e^B=e^{A+B}=e^B\,e^A. \tag{4.44}$$

b) Let $J=\begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$ be a Jordan block. Calculate e^{Jt} explicitly (Hint: use a)).

Exercise 4.4. Consider the linear ODE system

$$\dot{\boldsymbol{x}}=A(t)\,\boldsymbol{x}, \qquad \boldsymbol{x}(0)=\boldsymbol{x}_0. \tag{4.45}$$

Explain why in general

$$\exp\left[\int_0^t A(s)\,\mathrm{d}x\right]\boldsymbol{x}_0 \tag{4.46}$$

is not a solution.

4.1. Stationary solutions and stability analysis.

4.1.1. Evolution equations.

For evolutionary PDEs, in particular the autonomous type

$$u_t - L u = F(u) \tag{4.47}$$

the stationary solutions solve

$$-L u = F(u). \tag{4.48}$$

We can study its stability around these stationary solutions by turning it into an linear ODE system of infinite dimension through eigenfunction expansion.

Consider the following problem

$$u_t - u_{xx} = \lambda\, u\, (1 - u^2), \qquad u(0, t) = 0, u(\pi, t) = 0. \tag{4.49}$$

First we try to find stationary solution $u = u(x)$. These solutions solve

$$u_{xx} = \lambda\, u\, (1 - u^2) \tag{4.50}$$

which is still too hard to solve[4.1] so we settle with constant solutions:

$$u = 0, 1, -1. \tag{4.51}$$

- Around 0. The linearization is achieved by setting $u = 0 + \varepsilon\, w$ and discard all the terms of order higher than ε. Substituting $u = \varepsilon\, w$ into the equation we obtain

  $$\varepsilon\, w_t - \varepsilon\, w_{xx} = \lambda\, \varepsilon\, w\, (1 - \varepsilon^2\, w^2) \tag{4.52}$$

  which gives the linearized equation

  $$w_t - w_{xx} = \lambda\, w. \tag{4.53}$$

  Now recall that the eigenfunctions for this problem are

  $$\sin\,(n\, x). \tag{4.54}$$

  So write

  $$w(x, t) = \sum_{n=1}^{\infty} w_n(t) \sin\,(n\, x) \tag{4.55}$$

  and substitute into the equation, we have

  $$\dot{w}_n = (\lambda - n^2)\, w_n \tag{4.56}$$

  Thus if $\lambda \leqslant 1$, then the 0 solution is stable.

  However if $\lambda > 1$, some w_n will be increasing and eventually the neglected term $\lambda\, \varepsilon^2\, w^3$ may not be small compared to the linear terms anymore.

- Around 1. We set $u = 1 + \varepsilon\, w$ and substitute into the equation to reach

  $$\varepsilon\, w_t - \varepsilon\, w_{xx} = \lambda\, (1 + \varepsilon\, w)\, (-\varepsilon\, w)\, (2 + \varepsilon\, w) \tag{4.57}$$

  which gives

  $$w_t - w_{xx} = -2\, \lambda\, w. \tag{4.58}$$

4.1. The study of the higher dimension analog $\triangle u = u\, (1 - u^2)$ is an active research area as of now.

This leads to

$$\dot{w}_n = (-2\lambda - n^2)\, w_n \tag{4.59}$$

which gives stability for $\lambda > 0$.

Remark 4.6. If we consider constant solutions $u = u(t)$, then the equation becomes an ODE

$$\dot{u} = \lambda\, u\, (1 - u^2) \tag{4.60}$$

which can be easily analyzed.

4.1.2. Normal modes analysis.

For more general problems, it may not be possible or easy to figure out the set of eigenfunctions to use. In these cases, the method of "normal modes analysis" is very useful.

Consider a linear PDE with two variables x, t. Assume that t plays the role of "time" and takes values in $(0, \infty)$. Then it is natural to apply Fourier transform in x and Laplace transform in t.

$$a(k, \lambda) = c \int_0^\infty e^{-\lambda t} \left[\int_{-\infty}^\infty e^{-ikx}\, u(x, t)\, \mathrm{d}x \right] \mathrm{d}t \tag{4.61}$$

where c is the normalization factor whose exact value is not important here. From the inverse transform formulas we see that the solution is a linear combination of functions of the form

$$a(k, \lambda) \exp\,[i\, k\, x + \lambda\, t]. \tag{4.62}$$

Now after the transform the equation becomes an algebraic equation involving k, λ. Solving this equation would gives a relation $\lambda = \lambda(k)$. Therefore the solution is a combination of "normal modes":

$$a(k) \exp\,[i\, k\, x + \lambda(k)\, t]. \tag{4.63}$$

The idea of normal modes analysis is then to simply require

$$u(x, t) = a(k) \exp\,[i\, k\, x + \lambda(k)\, t] \tag{4.64}$$

to solve the equation and try to understand stability by studying the growth/decay of these "normal mode solutions".

Note that, in (4.64), $k \in \mathbb{R}$ but both $a(k)$ and $\lambda(k)$ may be complex.

Let's consider the linear second order equation

$$A\, u_{xx} + 2\, B\, u_{xt} + C\, u_{tt} + D\, u_x + E\, u_t + F u = 0. \tag{4.65}$$

It is clear that the only constant solution is $u = 0$.[4.2]

We consider a "normal mode" type solution:

$$u(x, t) = a(k) \exp\,[i\, k\, x + \lambda(k)\, t]. \tag{4.66}$$

Substituting (4.66) into the equation we obtain

$$C\, \lambda^2 + (2\, i\, B\, k + E)\, \lambda + (-A\, k^2 + i\, D\, k + F) = 0. \tag{4.67}$$

This is a quadratic equation and can be solved to give $\lambda = \lambda(k)$.

4.2. Note that we only discuss the homogeneous equation here. Since if the "forcing" term G is present, then it is not possible to have a constant solution anymore.

Recalling (4.66) we see that

$$|u(x,t)| = |a(k)| \exp\left[\Re\lambda(k)\, t\right]. \tag{4.68}$$

If we consider the behavior of this solution when t increases from 0, then clearly we should study the number

$$\Omega = \sup_{k\in\mathbb{R}} \left[\Re\lambda(k)\right]. \tag{4.69}$$

Theorem 4.7.

- *If $\Omega = +\infty$, then the problem is not well-posed. In the sense that the dependence of the solution to the initial condition is not continuous. In other words, the solution grows arbitrarily fast in arbitrarily short time.*

- *If $\Omega \in (0, \infty)$, then the problem is well-posed, but the equation is "unstable", in the sense that solutions grow and becomes unbounded as $t \longrightarrow \infty$.*

- *If $\Omega \in (-\infty, 0)$, then the problem is well-posed and the solution decays to 0 as $t \longrightarrow \infty$. In this case we say the equation is asymptotically stable.*

- *If $\Omega \in (-\infty, 0]$, we say the equation is neutrally stable.*

Proof.

- The case $\Omega = +\infty$. Take

  $$u(x,0) = \frac{1}{\lambda^2(k)}\, e^{ikx}. \tag{4.70}$$

  Then we have

  $$u(x,t) = \frac{1}{\lambda^2(k)} \exp\left[i\,k\,x + \lambda(k)\, t\right] \tag{4.71}$$

  which gives

  $$u_t(x,0) = \frac{1}{\lambda(k)}\, e^{ikx}. \tag{4.72}$$

  We have

  $$|u(x,0)| = \frac{1}{|\lambda(k)^2|}, \quad |u_t(x,0)| = \frac{1}{|\lambda(k)|}, \qquad |u(x,t)| = \frac{1}{|\lambda(k)^2|} \exp\left[\Re\lambda(k)\, t\right]. \tag{4.73}$$

  If we can show that $|\lambda(k)| \leqslant C\, \Re\lambda(k)$ for some C, then $u(x,0), u_t(x,0)$ can be arbitrary small while $u(x,t)$ arbitrary large.

- The case $\Omega \in (0, \infty)$. The well-posedness part is hard to prove. But we can easily see that some solutions are unbounded: If $\Omega > 0$ then there is $k_0 \in \mathbb{R}$ such that $\Re\lambda(k_0) > 0$. Then

  $$|a(k_0)| \exp[\Re\lambda(k_0)\, t] \longrightarrow \infty \tag{4.74}$$

  as $t \longrightarrow \infty$.

- Similar arguments show that when $\Omega \in (-\infty, 0)$ the solutions tends to 0 as $t \longrightarrow \infty$.

□

Example 4.8. (Well/ill-posedness of elliptic, parabolic, hyperbolic equations) Consider the equation

$$u_{xx} + u_{tt} + \rho\, u = 0 \tag{4.75}$$

where ρ is a constant. Then we obtain

$$\lambda(k) = \pm\sqrt{k^2 - \rho}. \tag{4.76}$$

We see that $\Omega = +\infty$ no matter how large ρ is. Therefore the initial value problem is ill-posed.

If instead we consider the parabolic equation

$$\rho\, u_{xx} + u_t = 0 \tag{4.77}$$

then we have

$$\lambda(k) = \rho\, k^2 \tag{4.78}$$

which is ill-posed only when $\rho > 0$.

Finally we consider the hyperbolic equation

$$u_{tt} - u_{xx} + \rho\, u = 0 \tag{4.79}$$

which gives

$$\lambda(k) = \pm i\,\sqrt{k^2 + \rho} \tag{4.80}$$

We have $\Re\lambda(k) = 0$ except for may be finite many k's. Thus $\Omega < +\infty$ and the Cauchy problem is well-posed.

Example 4.9. Consider the equation

$$u_{xx} + 2\, u_{xt} + u_{tt} = 0 \tag{4.81}$$

which gives

$$\lambda(k) = -i\, k \Longrightarrow \Omega = 0. \tag{4.82}$$

So the problem is neutrally stable.

In fact this equation can be explicitly solved:

$$u(x, t) = a\, e^{ik(x-t)} + b\, t\, e^{ik(x-t)} \tag{4.83}$$

So we see that in the neutrally stable case the solution still may be unbounded.

Example 4.10. Consider the Telegrapher's equation

$$u_{tt} - \gamma^2\, u_{xx} + 2\, \mu\, u_t = 0. \tag{4.84}$$

This gives

$$\lambda(k) = -\mu \pm \sqrt{\mu^2 - \gamma^2\, k^2}. \tag{4.85}$$

Since $\mu > 0$ we see that $\lambda(k) \leqslant 0$. But no matter what μ is, we have $\lambda(0) = 0$ so $\Omega = 0$. The equation is neutrally stable.

References.

- Erich Zauderer, "Partial Differential Equations of Applied Mathematics", 2ed, §3.5, §4.7.

Exercises.

Exercise 4.5. Analyze the well/ill-posedness of

$$u_{xx} + u_{tt} + u_t = 0 \tag{4.86}$$

using normal modes analysis.

4.2. Self-similar solutions.

Consider a PDE involving two variables x, t. A self-similar solution would be of the form

$$u(x,t) = \Phi(\xi) \tag{4.87}$$

where $\xi(x,t)$ usually is of the form $x^a t^b$.

4.2.1. Simple examples.

In simple cases the form of $\xi(x,t)$ can be guessed.

Example 4.11. (Rarefaction wave) We consider the scalar conservation law

$$u_t + f(u)_x = 0 \tag{4.88}$$

and try to find self-similar solutions of the form

$$u(x,t) = U(\xi), \qquad \xi = \frac{x}{t}. \tag{4.89}$$

Substituting this ansatz into the equation we have

$$-\xi U' + f'(U)\, U' = 0 \implies f'(U) = \xi \Longrightarrow U = r(\xi) \tag{4.90}$$

where r is the inverse function of f'.

In other words

$$u(x,t) = r\left(\frac{x}{t}\right). \tag{4.91}$$

Remark 4.12. One can "guess" the form of the self-similar ansatz $\xi = \frac{x}{t}$ through dimensional analysis. Consider the change of variables $x = \lambda x', t = \mu t'$. Let $u'(x', t') = u(x,t)$. Then

$$u_t = \mu^{-1} u'_{t'}, \qquad f(u)_x = \lambda^{-1} f(u)_{x'}. \tag{4.92}$$

We see that the equation does not change as long as $\lambda = \mu$. In this case the ratio x/t does not change. Therefore we should try using x/t as the new variable.

Example 4.13. (Porous medium) The porous medium equation reads

$$u_t - \triangle(u^\gamma) = 0 \tag{4.93}$$

with $u \geqslant 0$ and $\gamma > 1$.

We look for a solution having the form

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right). \tag{4.94}$$

Substituting into the equation we obtain

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot Dv(y) + t^{-(\alpha\gamma+2\beta)} \triangle(v^\gamma)(y) = 0. \tag{4.95}$$

Since the idea of introducing the self-similar ansatz is to simplify the equation (by reducing the number of variables), we try to cancel the t terms, by setting

$$\alpha + 1 = \alpha\gamma + 2\beta. \tag{4.96}$$

Then the equation reduces to

$$\alpha v + \beta y \cdot Dv + \triangle(v^\gamma) = 0. \tag{4.97}$$

This is still a PDE with n variables. To be able to solve it, we consider only the radial symmetric solution

$$v(y) = w(|y|). \tag{4.98}$$

Then the equation for w is

$$\alpha w + \beta r w' + (w^\gamma)'' + \frac{n-1}{r} (w^\gamma)' = 0. \tag{4.99}$$

Now if we further set

$$\alpha = n\,\beta, \tag{4.100}$$

we have (after multiplying the equation by r^{n-1})

$$(r^{n-1}\,(w^{\gamma})')' + \beta\,(r^n\,w)' = 0 \implies r^{n-1}(w^{\gamma})' + \beta\,r^n\,w = a. \tag{4.101}$$

To settle a, we assume that w decays at infinity, that is $w, w' \longrightarrow 0$ as $r \longrightarrow \infty$. This leads to $a = 0$. Therefore

$$(w^{\gamma})' = -\beta\,r\,w. \tag{4.102}$$

This gives

$$-\beta\,r\,w = \gamma\,w^{\gamma-1}\,w' \implies -\beta\,r = \gamma\,w^{\gamma-2}\,w' \implies (w^{\gamma-1})' = -\frac{\gamma-1}{\gamma}\,\beta\,r. \tag{4.103}$$

consequently

$$w^{\gamma-1} = b - \frac{\gamma-1}{2\,\gamma}\,\beta\,r^2 \tag{4.104}$$

which is

$$w = \max\left\{\left(b - \frac{\gamma-1}{\gamma}\,\beta\,r^2\right)^{\frac{1}{\gamma-1}}, 0\right\}. \tag{4.105}$$

Finally, returning to the original variables, we have the *Barenblatt solution*

$$u(x,t) = \left\{\frac{1}{t^{\alpha}}\left(b - \frac{\gamma-1}{2\,\gamma}\,\beta\,\frac{|x|^2}{t^{2\beta}}\right)^{\frac{1}{\gamma-1}}, 0\right\}. \tag{4.106}$$

4.2.2. Dimension analysis.

For more complicated problems, dimension analysis can be used to obtain the correct form of self-similarity.

The rationale behind dimension analysis is the understanding that physics should be invariant under a change of the units of measurement. For example, consider a simple pendulum of length l. We would like to know its period θ. Write

$$\theta = u(l, m, g) \tag{4.107}$$

where l is the length of the pendulum, m is the mass of the bob, and g is the gravity acceleration. Now if we decrease the unit of length by a factor L, the unit of mass by a factor M and the unit of time by a factor T, then we should have

$$T\theta = u\left(L\,l, M\,m, \frac{L}{T^2}\,g\right) \tag{4.108}$$

This means

$$u(l, m, g) = T^{-1}\,u\left(L\,l, M\,m, \frac{L}{T^2}\,g\right) \tag{4.109}$$

for all L, M, T. Taking $L = T = 1$ shows that u is independent of m, so we have

$$u(l, g) = T^{-1}\,u\left(L\,l, \frac{L}{T^2}\,g\right). \tag{4.110}$$

Now taking $T = 1$, we see that $u = \Phi(\xi)$ where $\xi = \frac{l}{g}$. This gives

$$\Phi(\xi) = T^{-1}\,\Phi(T^2\,\xi). \tag{4.111}$$

To find Φ we take $\frac{\mathrm{d}}{\mathrm{d}T}$:

$$0 = -\frac{1}{T^2}\,\Phi(T^2\,\xi) + 2\,\Phi'(T^2\,\xi). \tag{4.112}$$

Now set $T = 1$:

$$2\,\Phi' = \Phi \Longrightarrow \Phi = c\,\sqrt{\xi}. \tag{4.113}$$

Thus finally we have

$$\theta = c\sqrt{l/g}. \tag{4.114}$$

The value c can be determined from experiments.

Example 4.14. (Ground-water mound) Suppose that in a porous stratum over an underlying horizontal impermeable bed, water of a finite volume V is supplied instantaneously through a well of very small radius (for the sake of modelling, say the radius is 0) at time $t=0$. Then the water will form a mound whose height is decreasing with time. Our task is to figure out the shape of this mound.

Let k, m be the permeability and porosity (the relative volume of the pores), and let ρ be the water density, μ the dynamic viscosity, g the gravity accleration.

We further introduce the parameters

$$\kappa = \frac{k\,\rho\,g}{2\,m\,\mu}, \qquad Q = \frac{V}{2\,\pi\,m}. \tag{4.115}$$

Since the well has zero radius, we can assume the solution only depends on the radius r and the time t, that is the shape of the mound is given by

$$h = h(r,t). \tag{4.116}$$

It turns out that, if we consider the function $u = \rho\,g\,h(r,t)$, then it satisfies the Boussinesq equation

$$u_t = a\,\frac{1}{r}\,(r\,(u^2)_r)_r, \tag{4.117}$$

where the factor

$$a = \frac{k}{2\,m\,\mu} = \frac{\kappa}{\rho\,g}. \tag{4.118}$$

The initial conditions and boundary conditions are

$$u(r,0) = 0 \qquad r \neq 0; \qquad 2\,\pi\,m\int_0^\infty r\,u(r,0)\,\mathrm{d}r = \rho\,g\,V; \tag{4.119}$$

$$u(\infty,t) = 0. \tag{4.120}$$

Our task now is to find the self-similar solution.

To do this, we first need to figure out what the self-similarity looks like. Applying dimensional analysis we find two dimensionless variables:

$$\Pi = \frac{u(a\,t)^{1/2}}{(\rho\,g\,Q)^{1/2}} = \frac{h\,(\kappa\,t)^{1/2}}{Q^{1/2}}; \qquad \Pi_1 = \frac{r}{(\rho\,g\,Q\,a\,t)^{1/4}} = \frac{r}{(Q\,\kappa\,t)^{1/4}} \tag{4.121}$$

where recall that

$$Q = \frac{V}{2\,\pi\,m}, \qquad \kappa = a\,\rho\,g. \tag{4.122}$$

This gives

$$u = \rho\,g\,h = \frac{\rho\,g\,Q^{1/2}}{(\kappa\,t)^{1/2}}\,\Phi(\xi), \qquad \xi = \Pi_1 = \frac{r}{(Q\,\kappa\,t)^{1/4}}. \tag{4.123}$$

Substituting into (4.117) we obtain the equation for Φ as

$$\frac{\mathrm{d}^2(\Phi^2)}{\mathrm{d}\xi^2} + \frac{1}{\xi}\,\frac{\mathrm{d}(\Phi^2)}{\mathrm{d}\xi} + \frac{\xi}{4}\,\frac{\mathrm{d}\Phi}{\mathrm{d}\xi} + \frac{\Phi}{2} = 0. \tag{4.124}$$

To solve it, notice that if we multiply the equation by ξ, we get

$$\xi\,\frac{\mathrm{d}^2(\Phi^2)}{\mathrm{d}\xi^2} + \frac{\mathrm{d}(\Phi^2)}{\mathrm{d}\xi} + \frac{\xi^2}{4}\,\frac{\mathrm{d}\Phi}{\mathrm{d}\xi} + \frac{\xi}{2}\,\Phi = 0 \tag{4.125}$$

which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi\frac{\mathrm{d}(\Phi^2)}{\mathrm{d}\xi} + \frac{\xi^2}{4}\Phi\right] = 0. \tag{4.126}$$

Integrating, we have

$$\xi \frac{\mathrm{d}(\Phi^2)}{\mathrm{d}\xi} + \frac{\xi^2}{4}\Phi = C_1 \tag{4.127}$$

If we assume that Φ as well as its derivatives all tend to 0 as $r \longrightarrow \infty$ with t fixed (which becomes $\xi \longrightarrow \infty$) then clearly $C_1 = 0$. The equation simplifies to (assuming $\Phi \neq 0$)

$$\Phi' + \frac{\xi}{8} = 0 \Longrightarrow \Phi = (\xi_f^2 - \xi^2)/16 \tag{4.128}$$

for some constant ξ_f. Similar to the previous example, we impose the decay condition by "glueing" this Φ with the trivial solution $\Phi = 0$:

$$\Phi = \begin{cases} (\xi_f^2 - \xi^2)/16 & \xi \leqslant \xi_f \\ 0 & \xi \geqslant \xi_f \end{cases} \tag{4.129}$$

Here the constant ξ_f can be determined using the initial condition (4.119):

$$V = 2\,\pi\, m \int_0^\infty r\, h(r,t)\,\mathrm{d}r = \frac{Q^{1/2}\left[\, Q^{1/2}\,(\kappa\, t)^{1/2}\right] 2\,\pi\, m}{(\kappa\, t)^{1/2}} \int_0^{\xi_f} \xi\, \Phi(\xi)\,\mathrm{d}\xi = V \int_0^{\xi_f} \xi\, \Phi(\xi)\,\mathrm{d}\xi. \tag{4.130}$$

This leads to

$$1 = \frac{\xi_f^4}{64} \Longrightarrow \xi_f = \sqrt{8}. \tag{4.131}$$

Summarizing, the solution is

$$\Phi = \begin{cases} (8 - \xi^2)/16 & \xi \leqslant \sqrt{8} \\ 0 & \xi \geqslant \sqrt{8} \end{cases}. \tag{4.132}$$

and the shape of the mound is given by

$$h(r,t) = \max\left\{0, \frac{Q^{1/2}}{(\kappa\, t)^{1/2}} \left[8 - \frac{r^2}{(Q\,\kappa\, t)^{1/2}}\right]\right\}. \tag{4.133}$$

References.

- G. I. Barenblatt, "Scaling, self-similarity, and intermediate asymptotics", Cambridge University Press, 1996. §0.1 - §0.4.
- L. C. Evans, "Partial Differential Equations", 1ed, 1998, §4.2.

Exercises.

Exercise 4.6. (Dimension function is always power law) The following justifies the restriction to functions of the form $L^\alpha M^\beta T^\gamma$ when doing dimensional analysis.

Consider a physical relation

$$a = \phi(L, M, T) \tag{4.134}$$

where L, M, T stands for length, mass, time measured using some original units. Now let L_1, M_1, T_1 and L_2, M_2, T_2 be the numerical values of length, mass, time under two different sets of units. The numercal values of a are a_1, a_2 respectively. So we have

$$a_1 = a\,\frac{\phi(L_1, M_1, T_1)}{\phi(L, M, T)}, \qquad a_2 = a\,\frac{\phi(L_2, M_2, T_2)}{\phi(L, M, T)}. \tag{4.135}$$

If we assume that there is no preferred units, then we have

$$\frac{a_2}{a_1} = \phi(L_2/L_1, M_2/M_1, T_2/T_1). \tag{4.136}$$

Combining the above we reach

$$\frac{\phi(L_2, M_2, T_2)}{\phi(L_1, M_1, T_1)} = \phi(L_2/L_1, M_2/M_1, T_2/T_1) \tag{4.137}$$

for all L_1, M_1, T_1 and L_2, M_2, T_2. Prove that ϕ must take the form

$$\phi(L, M, T) = L^\alpha\, M^\beta\, T^\gamma \tag{4.138}$$

for some constants α, β, γ.

4.3. Traveling waves.

Example 4.15. **(Solitons)** Consider the KdV equation

$$u_t + 6\,u\,u_x + u_{xxx} = 0. \tag{4.139}$$

We look for solutions of the form

$$u(x,t) = v(x - \sigma\,t). \tag{4.140}$$

Substituting into the equation we have

$$-\sigma\,v' + 6\,v\,v' + v''' = 0. \tag{4.141}$$

Integrating once

$$-\sigma\,v + 3\,v^2 + v'' = a. \tag{4.142}$$

To integrate further, we multiply the equation by v'

$$-\sigma\,v\,v' + 3\,v^2\,v' + v'\,v'' = 0 \implies \frac{(v')^2}{2} = -v^3 + \frac{\sigma}{2}\,v^2 + a\,v + b. \tag{4.143}$$

We try to find solutions which decay to 0, this leads to $a = b = 0$. And the equation becomes

$$v' = \pm v\,(\sigma - 2\,v)^{1/2}. \tag{4.144}$$

Note that, if we let $s \to -s$, then v' changes sign but the RHS does not. Therefore once we solve the equation for one sign, setting $s \to -s$ would give us the solution to the equation with the opposite sign.

We take the minus sign. Let $v = \frac{\sigma}{2}\,w^2$. Then the equation for w is

$$w' = -\frac{\sqrt{\sigma}}{2}\,w\,(1 - w^2)^{1/2}. \tag{4.145}$$

Now inspired by $(1 - w^2)^{1/2}$ and keeping in mind that we need w decay at infinity, we set

$$w = \operatorname{sech}\theta = \frac{2}{e^{\theta} + e^{-\theta}}, \qquad \theta > 0. \tag{4.146}$$

We have

$$w' = -\frac{2\,(e^{\theta} - e^{-\theta})}{(e^{\theta} + e^{-\theta})^2}\,\theta', \;\; (1 - w^2)^{1/2} = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}. \tag{4.147}$$

Substituting into the equation we obtain

$$\theta' = \frac{\sqrt{\sigma}}{2} \implies \theta(s) = \frac{\sqrt{\sigma}}{2}\,(s - c) \tag{4.148}$$

where c is a constant.

Putting everything together we have

$$u(x,t) = \frac{\sigma}{2}\operatorname{sech}^2\!\left(\frac{\sqrt{\sigma}}{2}(x - \sigma\,t - c)\right). \tag{4.149}$$

Note that there is no restriction on σ. In other words, there exists traveling wave solutions for each speed σ.

Example 4.16. Consider the viscous conservation law

$$u_t + f(u)_x - a\,u_{xx} = 0 \qquad \text{in } \mathbb{R} \times (0, \infty) \tag{4.150}$$

where $a > 0$ and f is uniformly convex that is $f'' > \delta_0 > 0$ for some constant δ_0.

We look for travelling wave solutions. Set $u = v(x - \sigma\,t)$, the equation becomes

$$-\sigma\,v' + F(v)' - a\,v'' = 0. \tag{4.151}$$

Integrating, we have

$$-\sigma\,v + F(v) - a\,v' = -b. \tag{4.152}$$

This gives

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{F(v) - \sigma\,v + b}{a} \implies \frac{\mathrm{d}s}{\mathrm{d}v} = \frac{a}{F(v) - \sigma\,v + b}. \tag{4.153}$$

Therefore v is defined implicitly by the formula

$$s = \int_c^{v(s)} \frac{a}{F(z) - \sigma z + b} \mathrm{d}z \qquad (s \in \mathbb{R}), \tag{4.154}$$

where b and c are constants.

It turns out that we can find a traveling wave satisfying

$$\lim_{s \to -\infty} v(s) = u_l, \qquad \lim_{s \to \infty} v(s) = u_r \tag{4.155}$$

for $u_l > u_r$, if and only if

$$\sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r}. \tag{4.156}$$

- "Only if". Let $s \to \pm\infty$, naturally we require $v' \to 0$. thus

  $$-\sigma u_l + F(u_l) = -b, \qquad -\sigma u_r + F(u_r) = -b. \tag{4.157}$$

  The conclusion then follows.

- "If". Let $\sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r}$ for some $u_l > u_r$. We try to show $\lim_{s \to -\infty} v(s) = u_l, \lim_{s \to \infty} v(s) = u_r$. First observe that for this σ, we have

  $$F(u_l) - \sigma u_l = F(u_r) - \sigma u_r. \tag{4.158}$$

  Setting this value to be $-b$, we have

  $$F(u_l) - \sigma u_l + b = F(u_r) - \sigma u_r + b = 0. \tag{4.159}$$

  Next recall that F is uniformly convex, which means $F(z) - \sigma z + b$ can have at most two distinct roots. Therefore u_l and u_r are the only ones.

  Again due to the uniform convexity of F, it must be that $F > 0$ for $v < u_r$ and $v > u_l$, $F < 0$ for $v \in (u_r, u_l)$. We also have

  $$[F(z) - \sigma z + b]' = F'(z) - \sigma \tag{4.160}$$

  which is positive at u_l and negative at u_r.

  Putting these together, we have

  $$\int_c^{u_l} \frac{a}{F(z) - \sigma z + b} \mathrm{d}z = -\infty, \qquad \int_c^{u_r} \frac{a}{F(z) - \sigma z + b} \mathrm{d}z = \infty \tag{4.161}$$

  which means exactly

  $$\lim_{s \to -\infty} v(s) = u_l, \qquad \lim_{s \to \infty} v(s) = u_r \tag{4.162}$$

Finally we study the limiting behavior of $a \longrightarrow 0$. We will show that

> If we denote the above traveling wave solution of (4.150) u^a, with $u^a(0,0) = \frac{u_l + u_r}{2}$, then as $a \searrow 0$, u^a converges to the shock solution.

To see this, first note that, $v^a(s)$ cannot "cross" u_l or u_r. In other words, either $v^a(s) \geqslant u_l$, or $v^a(s) \leqslant u_r$, or $v^a(s) \in [u_r, u_l]$. To see this, assume the contrary. Wlog assume v^a has values above and below u_l. Then as $v^a \to u_l$ as $s \to -\infty$, there is s_0 such that v^a reaches maximum $v^a_{\max} > u_l$. At this point, we have $(v^a)' = 0$ and therefore $v_{\max}$ solves

$$-\sigma z + F(z) = -b. \tag{4.163}$$

But as $-\sigma z + F(z)$ is uniformly convex (thus strictly convex), there can be at most two solutions. As u_l and u_r already solve it, we obtain contradiction.[4.3]

4.3. From this it is clearly see that in the condition $u^a(0,0) = \frac{u_l + u_r}{2}$, the RHS can be replaced by any value in (u_r, u_l), as the purpose is just to restrict all v in $[u_l, u_r]$.

Next we show that, in fact for all s finite, $v(s) \in (u_r, u_l)$. To see this, study the formula

$$s = \int_c^{v(s)} \frac{\varepsilon}{F(z) - \sigma z + b} \, \mathrm{d}z. \tag{4.164}$$

As F is strictly convex, the behavior of the denominator close to u_l and u_r is like $(z - z_0)^{-1}$. And thus if $v(s) = u_l$ or u_r, necessarily $s = \mp\infty$.

Finally, fix any $s \neq 0$ finite, we have

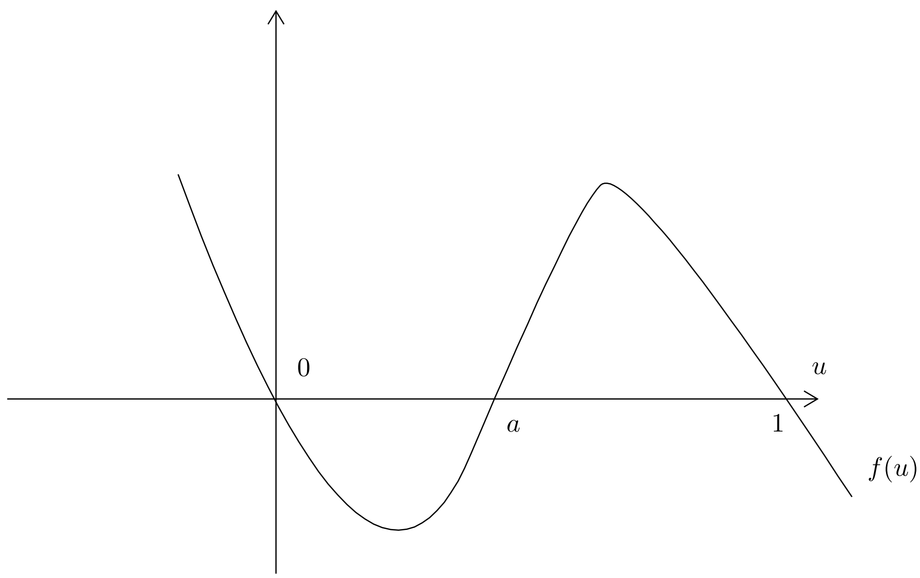
$$\frac{s}{a} = \int_c^{v(s)} \frac{1}{F(z) - \sigma z + b} \, \mathrm{d}z. \tag{4.165}$$

When $a \searrow 0$, the LHS $\to \mp\infty$, consequently $v(s) \to u_l$ or u_r at every s. Thus we see that as $a \searrow 0$, v^a converges to $v = \begin{cases} u_l & s < 0 \\ u_r & x > 0 \end{cases}$ at every $s \neq 0$.

Example 4.17. (A bistable equation) Consider the reaction-diffusion equation

$$u_t - u_{xx} = f(u) \tag{4.166}$$

with f having a "cubic-like" shape



We further assume that

$$\int_0^1 f(z) \, \mathrm{d}z > 0. \tag{4.167}$$

If we omit the u_{xx} term, then we see that 0 and 1 are stable fixed points, while a is unstable. Thus we try to find a traveling wave solution satisfying $u(-\infty) = 0, u(+\infty) = 1$.

Setting $u(x,t) = v(x - \sigma t)$, the equation becomes

$$-\sigma v' - v'' = f(v) \tag{4.168}$$

Our task is to show that one can find σ such that the desired solution exists. Thus this is a nonlinear "eigenvalue" problem.

Now let $w = v'$, we have the following ODE system

$$v' = w, \tag{4.169}$$

$$w' = -\sigma w - f(v). \tag{4.170}$$

This system has three fixed points: $(0,0),(a,0),(1,0)$. Linearize at any fixed point we have

$$v' = w \tag{4.171}$$
$$w' = -f'v - \sigma w \tag{4.172}$$

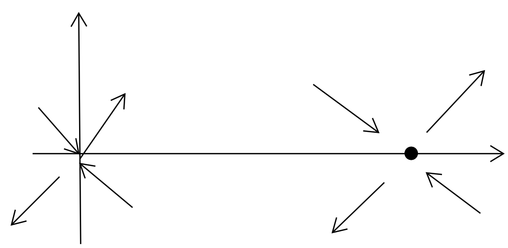
whose eigenvalue equation is

$$\lambda^2 + \sigma\lambda + f' = 0. \tag{4.173}$$

with eigenvectors

$$w = \lambda v. \tag{4.174}$$

Now at 0 and 1, we have $f' < 0$ which means we have one positive eigenvalue and one negative eigenvalue with corresponding eigenvectors as follows:



On the other hand, at $(a,0)$ we have two cases:

i. $\sigma > 0$. In this case both eigenvalues are positive.

ii. $\sigma < 0$. In this case both eigenvalues are negative.

The two cases are illustrated as follows:

From this analysis we know that there is one trajectory starting from $(0,0)$ into the first quadrant. Now the task is to show the existence of σ such that this trajectory ends at $(1,0)$.

Notice that if we set

$$E(v,w) := \frac{w^2}{2} + \int_0^v f(z)\,\mathrm{d}z. \tag{4.175}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}E = -\sigma w^2 \begin{cases} \leqslant 0 & \sigma > 0 \\ \geqslant 0 & \sigma < 0 \end{cases}. \tag{4.176}$$

Drawing the levelsets of $E(v,w)$, we see that if $\sigma > 0$, then the solution $(v(t),w(t))$ is contained inside the levelset $E(v,w) = 0$ and is not possible to reach $(1,0)$.

Now we consider the case $\sigma < 0$. In this case we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E \geqslant 0 \tag{4.177}$$

with "=" only when $w = 0$ or $\sigma = 0$.

Now recall the equation for v which is

$$v' = w. \tag{4.178}$$

If the solution curve stays outside the level set $E(v,w)=0$, we have $w>0$ therefore $v'>0$. This implies that $(v(t),w(t))$ has to cross the vertical line $L: v=a+\varepsilon$ for some $\varepsilon>0$ which may be very small.

On the other hand, similar analysis for $(v(t),w(t))$ starting from $(1,0)$ (with t decreasing) shows that it also has to reach L.

Now our task is to show that there is $\sigma<0$ such that two curves – one from $(0,0)$ and the other from $(1,0)$ will meet at the same point on L which means they have to belong to the same solution. In the following we denote the intersection of the left curve with L by $w_L(\sigma)$ and that of the right curve with L by $w_R(\sigma)$. All we need to do is to show that there is $\sigma<0$ such that $w_L(\sigma)=w_R(\sigma)$. As w_L, w_R are clearly continuous functions of σ, we try to apply intermediate value theorem.

First notice that $w_L(0)<w_R(0)$ because the assumption $\int_0^1 f(z)\,\mathrm{d}z>0$.

On the other hand, we observe that as σ decreases, $w_L(\sigma)$ is increasing while $w_R(\sigma)$ is decreasing. More specifically, along any solution curve,

$$\frac{\mathrm{d}w}{\mathrm{d}v}=\frac{-\sigma\,w-f(v)}{w}=-\sigma-\frac{f(v)}{w}\geqslant-\sigma-\frac{C}{w} \tag{4.179}$$

where C is an upper bound for $f(v)$. From this we see that for any $\beta>0$ there is $\Sigma<0$ such that for all $\sigma<\Sigma$, $\frac{\mathrm{d}w}{\mathrm{d}v}>\beta$. It's clear now that by taking σ negative enough, we have $w_L(\sigma)>w_R(\sigma)$.

Summarizing the above, we see that there exists a travelling wave solution with $u(-\infty)=0, u(+\infty)=1$.

Remark 4.18. For $\sigma>0$ naturally one should assume $\int_0^1 f(z)\,\mathrm{d}z<0$. Does the proof still work?

References.

- L. C. Evans, "Partial Differential Equations", 1ed, 1998, §4.2.