3. Separation of Variables

3.0. Basics of the Method.

In this lecture we review the very basics of the method of separation of variables in 1D.

3.0.1. The method.

The idea is to write the solution as

$$u(x,t)=\sum_n X_n(x)\,T_n(t). \tag{3.1}$$

where $X_n(x)\,T_n(t)$ solves the equation and satisfies the boundary conditions (but not the initial condition(s)). We give a summary using heat equation here. Given equation

$$\frac{\partial u}{\partial t}=\beta\,\frac{\partial^2 u}{\partial x^2}+P(x,t), \qquad 0<x<L; \qquad u(x,0)=f(x), \qquad +\text{boundary conditions} \tag{3.2}$$

1. Require $X(x)\,T(t)$ to solve the homogeneous equation

$$\frac{\partial u}{\partial t}=\beta\,\frac{\partial^2 u}{\partial x^2} \tag{3.3}$$

   which leads to eigenvalue problem for X:

$$X''-K\,X=0+\text{boundary conditions}. \tag{3.4}$$

   Solve it to get X_n and K_n. Note that the natural range of n is **not** always $1,2,3,\ldots$

2. Expand

$$f(x)=\sum_n f_n\,X_n. \tag{3.5}$$

   Expand

$$P(x,t)=\sum_n p_n(t)\,X_n. \tag{3.6}$$

3. Solve

$$T_n'-\beta\,K_n\,T_n=p_n(t), \qquad T_n(0)=f_n \tag{3.7}$$

   to obtain T_n.

4. Write down the solution

$$u(x,t)=\sum_n T_n(t)\,X_n(x). \tag{3.8}$$

We understand that changes should be made when the equation is different.

3.0.2. Examples.

Example 3.1. (Simplest case) Solve

$$\frac{\partial u}{\partial t}=3\,\frac{\partial^2 u}{\partial x^2}, \ 0<x<\pi; \qquad u(0,t)=u(\pi,t)=0, \quad u(x,0)=\sin x-4\sin 3\,x. \tag{3.9}$$

Solution. We follow the procedure:

1. Obtain X_n;

   a. Separate variables. Recall that $X(x)\,T(t)$ must solve the equation. This gives

$$X(x)\,T'(t)=3\,X''(x)\,T(t)\Longrightarrow\frac{T'(t)}{T(t)}=3\,\frac{X''(x)}{X(x)}. \tag{3.10}$$

   So we have the equations for T and X:

$$T'(t)-3\,K\,T(t)=0; \qquad X''(x)-K\,X(x)=0. \tag{3.11}$$

b. Boundary conditions for the X equation:

$$u(0,t)=0 \Longrightarrow X(0)=0; \qquad u(\pi,t)=0 \Longrightarrow X(\pi)=0. \tag{3.12}$$

c. Solve for X_n: (It's important to be able to solve the eigenvalue problem from scratch. You may be required to write down every detail – instead of just write $X_n=\cdots$ from memory – in the exam.)

The eigenvalue problem is

$$X''-K\,X=0, \qquad X(0)=X(\pi)=0. \tag{3.13}$$

We discuss the three cases:

i. $K>0$. General solution is

$$X=C_1\,e^{\sqrt{K}\,x}+C_2\,e^{-\sqrt{K}\,x}. \tag{3.14}$$

Applying the initial conditions we conclude $C_1=C_2=0$.

ii. $K=0$. General solution is

$$X=C_1+C_2\,x. \tag{3.15}$$

Applying the initial conditions we conclude $C_1=C_2=0$.

iii. $K<0$. General solution is

$$X=C_1\cos\left(\sqrt{-K}\,x\right)+C_2\sin\left(\sqrt{-K}\,x\right). \tag{3.16}$$

Applying initial conditions we conclude that $C_1=C_2=0$ unless $K=-n^2$.

So the eigenvalues are

$$K_n=-n^2, \qquad n=1,2,3,\ldots \tag{3.17}$$

with eigenfunctions

$$C\sin\left(n\,x\right). \tag{3.18}$$

Thus we take[3.1]

$$X_n=\sin\left(n\,x\right). \tag{3.19}$$

2. Expand $f(x)$;

$$f(x)=\sin x-4\sin 3\,x=X_1-4\,X_3. \tag{3.20}$$

Thus we have

$$f_1=1,\ f_3=-4, \text{all other } f_n=0. \tag{3.21}$$

3. Solve T_n;

Now we have both equation and initial condition for T_n:

$$T_n'-3\,K_n\,T_n=0, \qquad T_n(0)=f_n. \tag{3.22}$$

This gives

$$T_n(t)=T_n(0)\,e^{3K_nt}=f_n\,e^{3K_nt}. \tag{3.23}$$

Clearly if $f_n=0$ then $T_n=0$. So the only nonzero ones are

$$T_1(t)=f_1\,e^{3K_1t}=1\cdot e^{3(-1)t}=e^{-3t}; \tag{3.24}$$

$$T_3(t)=f_3\,e^{3K_3t}=-4\,e^{-27t}. \tag{3.25}$$

4. Write down solution.

We have

$$u(x,t)=\sum_{n=1}^{\infty}T_n(t)\,X_n(x)=e^{-3t}\sin x-4\,e^{-27t}\sin 3\,x. \tag{3.26}$$

3.1. This is just for convenience. There is some freedom in choosing X_n here. For example, taking $X_n=2\sin n\,x$ is also OK.

Example 3.2. (Equation with Source) Solve

$$\frac{\partial u}{\partial t} = 3\,\frac{\partial^2 u}{\partial x^2} + e^t \sin x,\ \ 0 < x < \pi; \qquad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = \sin x - 4\sin 3\,x. \tag{3.27}$$

Solution. We try to solve this problem using the same X_n's. Recall that we assume

$$u(x,t) = \sum T_n(t)\, X_n(x) \tag{3.28}$$

with $X_n'' - 3\,K_n\,X_n = 0$. Substitute this $u(x,t)$ into the equation we reach

$$\sum_{n=1}^{\infty} [T_n' - 3\,K_n\,T_n]\,X_n = t\sin x = e^t\,X_1. \tag{3.29}$$

Absorbing the $t\,X_1$ term into the left hand side we reach

$$[T_1' - 3\,K_1\,T_1 - e^t]\,X_1 + \sum_{n=2}^{\infty} [T_n' - 3\,K_n\,T_n]\,X_n = 0 \tag{3.30}$$

This leads to

$$T_1' - 3\,K_1\,T_1 - e^t = 0; \qquad T_n' - 3\,K_n\,T_n = 0, n > 1. \tag{3.31}$$

- Solve T_1 from

  $$T_1' - 3\,K_1\,T_1 - e^t = 0, \qquad T_1(0) = 1. \tag{3.32}$$

  As $K_1 = -1$ the above is

  $$T_1' + 3\,T = e^t, \qquad T_1(0) = 1 \tag{3.33}$$

  The equation is linear with general solution

  $$T_1(t) = \frac{1}{4}\,e^t + C\,e^{-3t}. \tag{3.34}$$

  Using the initial condition we reach

  $$C = \frac{3}{4} \Longrightarrow T_1(t) = \frac{1}{4}\,e^t + \frac{3}{4}\,e^{-3t}. \tag{3.35}$$

- $T_n, n > 1$ remains the same as in the previous example:

  $$T_3(t) = -4\,e^{-27t}, \quad T_n(t) = 0 \text{ for } n \neq 3. \tag{3.36}$$

So finally

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\,X_n(x) = \left(\frac{1}{4}\,e^t + \frac{3}{4}\,e^{-3t}\right)\sin x - 4\,e^{-27t}\sin 3\,x. \tag{3.37}$$

Example 3.3. (Dirichlet boundary condition) Find the solution to the heat flow problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= 5\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi,\ \ t > 0 && (3.38)\\ u(0,t) = u(\pi,t) &= 0, \qquad\qquad t > 0 && (3.39)\\ u(x,0) &= f(x) = 1 - \cos 2\,x, \qquad 0 < x < \pi. && (3.40) \end{aligned}$$

Solution. We use separation of variables.

1. Separate the variables.

   Write $u = X(x)\,T(t)$, the equation becomes

   $$T'\,X = 5\,X''\,T \Longrightarrow \frac{T'}{5\,T} = \frac{X''}{X} = K. \tag{3.41}$$

   The equations for X is

   $$X'' - K\,X = 0, \qquad X(0) = X(\pi) = 0; \tag{3.42}$$

   The left hand side of the equation for T is

   $$T' - 5\,K\,T = \cdots. \tag{3.43}$$

2. Solve the eigenvalue problem

$$X'' - K\,X = 0, \qquad X(0) = X(\pi) = 0; \tag{3.44}$$

As we have solved it before, we omit the details (You need to include the details in the exam though). The eigenvalues are $-n^2$, $n = 1, 2, 3, \ldots$, and the corresponding eigenfunctions are

$$b_n \sin(n\,x), \qquad n = 1, 2, 3, \ldots \tag{3.45}$$

So $K_n = -n^2$, $X_n = \sin(n\,x)$.

3. Expand the initial condition. We have

$$1 - \cos 2\,x = u(x, 0) = \sum_{n=1}^{\infty} b_n \sin(n\,x). \tag{3.46}$$

Thus all we need to do is to find the Fourier sine series for $1 - \cos 2\,x$. As the interval is $[0, \pi]$ we have $T = \pi$. We compute for $n = 1, 2, 3, \ldots$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} (1 - \cos 2\,x)\sin(n\,x)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^{\pi} \sin(n\,x)\,\mathrm{d}x - \frac{2}{\pi}\int_0^{\pi} \sin(n\,x)\cos(2\,x)\,\mathrm{d}x \\ &= -\frac{2}{n\,\pi}\cos(n\,x)|_0^{\pi} - \frac{1}{\pi}\int [\sin(n+2)\,x + \sin(n-2)\,x]\,\mathrm{d}x \\ &= \frac{2}{n\,\pi}[1 - (-1)^n] - \frac{1}{\pi}\int_0^{\pi} \sin(n+2)\,x\,\mathrm{d}x - \frac{1}{\pi}\int_0^{\pi} \sin(n-2)\,x\,\mathrm{d}x. \end{aligned} \tag{3.47}$$

We evaluate

$$\int_0^{\pi} \sin(n+2)\,x\,\mathrm{d}x = -\frac{1}{n+2}\cos(n+2)\,x|_0^{\pi} = \frac{1 - (-1)^{n+2}}{n+2}. \tag{3.48}$$

For the last term, there are two cases.

- If $n = 2$, then $\sin(n-2)\,x = 0$ and

$$\int_0^{\pi} \sin(n-2)\,x\,\mathrm{d}x = 0. \tag{3.49}$$

- If $n \neq 2$, we compute

$$\int_0^{\pi} \sin(n-2)\,x\,\mathrm{d}x = -\frac{1}{n-2}\cos(n-2)\,x|_0^{\pi} = \frac{1 - (-1)^{n-2}}{n-2}. \tag{3.50}$$

Putting everything together, we have

$$b_n = \begin{cases} \left(\frac{2}{n} - \frac{1}{n+2}\right)\frac{1 - (-1)^n}{\pi} & n = 2 \\ \left(\frac{2}{n} - \frac{1}{n-2} - \frac{1}{n+2}\right)\frac{1 - (-1)^n}{\pi} & n \neq 2 \end{cases}. \tag{3.51}$$

Noticing that, when n is even, we have $1 - (-1)^n = 0$. Thus the above formula can be simplified by setting $n = 2\,k - 1$ to

$$b_{2k-1} = \left(\frac{2}{2\,k-1} - \frac{1}{2\,k-3} - \frac{1}{2\,k+1}\right)\frac{2}{\pi}, \qquad k = 1, 2, 3, \ldots \tag{3.52}$$

As $P(x, t) = 0$ here, the expansion is trivial:

$$0 = \sum_{n=1}^{\infty} 0 \sin n\,x. \tag{3.53}$$

4. Solve the equation for T. As $p_n(t) = 0$ for all n, T_n satisfies

$$T_n' + 5\,n^2\,T_n = 0 \Longrightarrow T_n = c_n\,e^{-5n^2 t}. \tag{3.54}$$

5. Putting everything together, the solution is given by

$$u(x,t)=\sum_{k=1}^{\infty}\left(\frac{2}{2\,k-1}-\frac{1}{2\,k-3}-\frac{1}{2\,k+1}\right)\frac{2}{\pi}\,e^{-5(2k-1)^2t}\sin\left((2\,k-1)\,x\right). \tag{3.55}$$

Example 3.4. (Neumann boundary condition) Solve

$$\begin{aligned}\frac{\partial u}{\partial t} &= 3\,\frac{\partial^2 u}{\partial x^2}, && 0<x<\pi, \quad t>0 && (3.56)\\ \frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(\pi,t) &= 0, && t>0, && (3.57)\\ u(x,0) &= x, && 0<x<\pi. && (3.58)\end{aligned}$$

Solution. We use separation of variables.

1. Separate the variables.

   Write $u(x,t)=X(x)\,T(t)$. Then the equation leads to

$$T'\,X=3\,X''\,T\Longrightarrow\frac{T'}{3\,T}=\frac{X''}{X}=K \tag{3.59}$$

   which gives

$$X''-K\,X=0, \qquad X'(0)=X'(\pi)=0, \tag{3.60}$$

   and

$$T'-3\,K\,T=\cdots. \tag{3.61}$$

   Note that $\cdots$ comes from the expansion of $P(x,t)$ (=0 for this particular problem).

   Now Solve the eigenvalue problem

$$X''-K\,X=0, \qquad X'(0)=X'(\pi)=0, \tag{3.62}$$

   We discuss the three cases.

   a. $K<0$. The general solution is

$$X=C_1\cos\left(\sqrt{-K}\,x\right)+C_2\sin\left(\sqrt{-K}\,x\right). \tag{3.63}$$

   We compute

$$X'=-\sqrt{-K}\,C_1\sin\left(\sqrt{-K}\,x\right)+\sqrt{-K}\,C_2\cos\left(\sqrt{-K}\,x\right). \tag{3.64}$$

   The boundary conditions then lead to

$$\begin{aligned}\sqrt{-K}\,C_2 &= 0, && (3.65)\\ -\sqrt{-K}\,C_1\sin\left(\sqrt{-K}\,\pi\right)+\sqrt{-K}\,C_2\cos\left(\sqrt{-K}\,\pi\right) &= 0. && (3.66)\end{aligned}$$

   Thus $C_2=0$, and $\sqrt{-K}=n$. Consequently the eigenvalues are

$$K_n=-n^2, \qquad n=1,2,3,\ldots \tag{3.67}$$

   with corresponding

$$X_n=\cos\left(n\,x\right). \tag{3.68}$$

   b. $K=0$. The general solution is

$$X=C_1+C_2\,x \tag{3.69}$$

   the boundary conditions then gives $C_2=0$ which means 0 is an eigenvalue and the corresponding eigenfunctions are $X_0=1$.

   c. $K>0$. The general solution is

$$X=C_1\,e^{\sqrt{K}\,x}+C_2\,e^{-\sqrt{K}\,x}. \tag{3.70}$$

   The boundary conditions leads to

$$\begin{aligned}\sqrt{K}\,C_1-\sqrt{K}\,C_2 &= 0 && (3.71)\\ \sqrt{K}\,e^{\sqrt{K}\,\pi}\,C_1-\sqrt{K}\,e^{-\sqrt{K}\,\pi}\,C_2 &= 0. && (3.72)\end{aligned}$$

Solving it gives $C_1 = C_2 = 0$. Therefore there is no positive eigenvalues.

Summarizing, the eigenvalues are

$$K_n = -n^2, \qquad n = 0, 1, 2, 3, \ldots \tag{3.73}$$

$$X_n = \cos(n\,x), \qquad n = 0, 1, 2, 3, \ldots$$

2. Expand $f(x)$. We have

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,x). \tag{3.74}$$

All we need to do is to find the cosine series for x: $0 < x < \pi$. We compute

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x\,\mathrm{d}x = \pi, \tag{3.75}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos(n\,x)\,\mathrm{d}x = \frac{2}{n\,\pi} \left[x \sin(n\,x)|_0^{\pi} - \int_0^{\pi} \sin(n\,x)\,\mathrm{d}x \right] = \frac{2\,[(-1)^n - 1]}{n^2\,\pi}. \tag{3.76}$$

Note that $(-1)^n - 1 = 0$ for all n even. Thus we have

$$a_{2k} = 0, \qquad a_{2k-1} = -\frac{4}{(2\,k-1)^2\,\pi}. \tag{3.77}$$

3. Solve the equation for T_n:

$$T_n' + 3\,n^2\,T_n = 0, T_n(0) = \begin{cases} \frac{\pi}{2} & n = 0 \\ 0 & n \text{ even} \\ -\frac{4}{n^2\,\pi} & n \text{ odd} \end{cases} \Longrightarrow T_n = \begin{cases} \frac{\pi}{2} & n = 0 \\ 0 & n \text{ even} \\ -\frac{4}{n^2\,\pi}\,e^{-3n^2 t} & n \text{ odd} \end{cases}. \tag{3.78}$$

4. Summarizing, we have

$$u(x,t) = \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4}{(2\,k-1)^2\,\pi}\,e^{-3(2k-1)^2 t} \cos((2\,k-1)\,x). \tag{3.79}$$

Example 3.5. (Other boundary conditions) Solve

$$\begin{aligned} u_t &= \kappa\,u_{xx} & 0 < x < l, t > 0 \quad (3.80)\\ u(x,0) &= f(x) & 0 \leqslant x \leqslant l, \quad (3.81)\\ u(0,t) &= 0 & t \geqslant 0, \quad (3.82)\\ u_x(l,t) &= -h\,u(l,t). & t \geqslant 0. \quad (3.83) \end{aligned}$$

Here $h > 0$.

Solution. Applying the method of separation of variables, we reach

$$X'' - \lambda\,X = 0, \qquad X(0) = 0, \quad h\,X(l) + X'(l) = 0. \tag{3.84}$$

We discuss the cases:

i. $\lambda > 0$. The general solution is

$$A\,e^{\sqrt{\lambda}x} + B\,e^{-\sqrt{\lambda}x}. \tag{3.85}$$

Now

$$X(0) = 0 \Longrightarrow A + B = 0 \tag{3.86}$$

$$h\,X(l) + X'(l) = 0 \Longrightarrow \left(h + \sqrt{\lambda}\right) e^{\sqrt{\lambda}l} A + \left(h - \sqrt{\lambda}\right) e^{-\sqrt{\lambda}l} B = 0. \tag{3.87}$$

The two equations can be written

$$\begin{pmatrix} 1 & 1 \\ \left(h + \sqrt{\lambda}\right) e^{\sqrt{\lambda}l} & \left(h - \sqrt{\lambda}\right) e^{-\sqrt{\lambda}l} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.88}$$

For the solution to be non-zero, we have to have

$$0=\det\begin{pmatrix} 1 & 1 \\ (h+\sqrt{\lambda})\,e^{\sqrt{\lambda}l} & (h-\sqrt{\lambda})\,e^{-\sqrt{\lambda}l} \end{pmatrix}=(h-\sqrt{\lambda})\,e^{-\sqrt{\lambda}l}-(h+\sqrt{\lambda})\,e^{\sqrt{\lambda}l}. \tag{3.89}$$

As $h>0$ and $\sqrt{\lambda}>0$, this is not possible.

ii. $\lambda=0$. The general solution is

$$A+B\,x \tag{3.90}$$

The boundary conditions lead to

$$A=0, \qquad h\,A+(h\,l+1)\,B=0 \implies A=B=0. \tag{3.91}$$

iii. $\lambda<0$. The general solution is

$$A\cos\left(\sqrt{-\lambda}\,x\right)+B\sin\left(\sqrt{-\lambda}\,x\right). \tag{3.92}$$

Now

$$X(0)=0 \iff A=0, \tag{3.93}$$

$$h\,X(l)+X'(l)=0 \iff h\sin\left(\sqrt{-\lambda}\,l\right)+\sqrt{-\lambda}\cos\left(\sqrt{-\lambda}\,l\right)=0. \tag{3.94}$$

Therefore the solution is of the form $X=\sin(p\,x)$ with p satisfying

$$\tan(p\,l)=-p/h. \tag{3.95}$$

It is easy to see that the solutions form an infinite series

$$0<p_1<p_2<\cdots<\cdots \tag{3.96}$$

Therefore our solution to the PDE can be written as

$$\sum_1^\infty b_n\,e^{-\kappa p_n^2 t}\sin(p_n\,x) \tag{3.97}$$

where b_n is determined by

$$f(x)=\sum_1^\infty b_n\sin(p_n\,x). \tag{3.98}$$

Now how should we determine b_n? And furthermore how can we know whether the infinite sum gives the solution – or equivalently whether similar properties as those hold for the Fourier series hold for our series with $\sin(p_n\,x)$? Keep in mind that it is not possible to obtain a formula for the p_ns.

We compute, for $n\neq m$,

$$\begin{aligned}\int_0^l \sin(p_n\,x)\sin(p_m\,x)\,\mathrm{d}x &= \frac{1}{2}\int_0^l [\cos(p_n-p_m)\,x-\cos(p_n+p_m)\,x]\,\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{\sin(p_n-p_m)\,l}{p_n-p_m}-\frac{\sin(p_n+p_m)\,l}{p_n+p_m}\right] \\ &= \frac{1}{2}\left[\frac{\sin(p_n\,l)\cos(p_m\,l)-\sin(p_m\,l)\cos(p_n\,l)}{p_n-p_m}\right. \\ &\quad \left.-\frac{\sin(p_n\,l)\cos(p_m\,l)+\sin(p_m\,l)\cos(p_n\,l)}{p_n+p_m}\right].\end{aligned} \tag{3.99}$$

Now using the fact that

$$h\sin(p_n\,l)+p_n\cos(p_n\,l)=0 \implies \sin(p_n\,l)=-\frac{p_n}{h}\cos(p_n\,l) \tag{3.100}$$

we have

$$\int_0^l \sin(p_n\,x)\sin(p_m\,x)\,\mathrm{d}x = 0. \tag{3.101}$$

Therefore we can determine b_n by

$$b_n = \frac{\int_0^l f(x) \sin(p_n x)\, \mathrm{d}x}{\int_0^l \sin^2(p_n x)\, \mathrm{d}x}. \tag{3.102}$$

Example 3.6. (Wave equation)

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= 4 \frac{\partial^2 u}{\partial x^2} + x\,t, \qquad 0 < x < \pi, \quad t > 0 && (3.103)\\
u(0,t) = 0 \quad &, \quad u(\pi,t) = 1 \qquad t > 0 && (3.104)\\
u(x,0) &= x, \qquad 0 < x < \pi, && (3.105)\\
\frac{\partial u}{\partial t}(x,0) &= 1, \qquad 0 < x < \pi. && (3.106)
\end{aligned}$$

Remark. Note that trying to find a "steady-state solution" first would fail this time: No $w(x)$ can possibly satisfy

$$4\, w_{xx} + x\, t = 0, \qquad w(0) = 0, w(\pi) = 1. \tag{3.107}$$

Furthermore, any effort of finding $w(x,t)$ such that

$$w_{tt} = 4\, w_{xx} + x\, t, \qquad w(0,t) = 0, \quad w(\pi,t) = 1 \tag{3.108}$$

in hope of $v = u - w$ satisfying the simple system

$$\begin{aligned}
\frac{\partial^2 v}{\partial t^2} &= 4 \frac{\partial^2 v}{\partial x^2}, \qquad 0 < x < \pi, \quad t > 0 && (3.109)\\
v(0,t) = 0 \quad &, \quad v(\pi,t) = 0 \qquad t > 0 && (3.110)\\
u(x,0) &= x - w(x,0), \qquad 0 < x < \pi, && (3.111)\\
\frac{\partial u}{\partial t}(x,0) &= 1 - \frac{\partial w}{\partial t}(x,0), \qquad 0 < x < \pi. && (3.112)
\end{aligned}$$

does not make much sense as solving the w equation is not really easier than solving the original u equation.

In summary, in such general case, it is not possible to take care of the boundary conditions and the source term in one single step. They have to be dealt with separately.

Solution.

- Step 0. Take care of boundary conditions.

  We find $w(x)$ satisfying $w_{xx} = 0$, $w(0) = 0$, $w(\pi) = 1$. This is easy: $w(x) = x/\pi$. Now set $v = u - w$, we reach

$$\begin{aligned}
\frac{\partial^2 v}{\partial t^2} &= 4 \frac{\partial^2 v}{\partial x^2} + x\,t, \qquad 0 < x < \pi, \quad t > 0 && (3.113)\\
v(0,t) = 0 \quad &, \quad v(\pi,t) = 0 \qquad t > 0 && (3.114)\\
u(x,0) &= x - x/\pi, \qquad 0 < x < \pi, && (3.115)\\
\frac{\partial u}{\partial t}(x,0) &= 1, \qquad 0 < x < \pi. && (3.116)
\end{aligned}$$

- Step 1. Find the eigenvalue problem and solve it.

  Applying separation of variables, we found out that the eigenvalue problem is

$$X'' - \lambda\, X = 0, \qquad X(0) = X(\pi) = 0 \tag{3.117}$$

  which leads to (*details omitted due to having been done several times before*)

$$\lambda_n = -n^2, \qquad X_n = A_n \sin(n\, x), \qquad n = 1, 2, 3, \ldots \tag{3.118}$$

- Step 2. Find out equations for T_n.

  We write

$$v(x,t) = \sum_{n=1}^{\infty} T_n\, X_n = \sum_{n=1}^{\infty} T_n(t) \sin(n\, x) \tag{3.119}$$

Note that the arbitrary constant A_n has been "absorbed" into $T_n(t)$.

Substitute into the equation:

$$\frac{\partial^2 v}{\partial t^2} = \sum_{n=1}^{\infty} T_n''(t) \sin(n\,x); \tag{3.120}$$

$$\frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty} T_n(t)\,[\sin(n\,x)]'' = -\sum_{n=1}^{\infty} n^2\,T_n(t) \sin(n\,x); \tag{3.121}$$

Therefore the equation becomes

$$x\,t = \frac{\partial^2 v}{\partial t^2} - 4\,\frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty} [T_n'' + 4\,n^2\,T_n] \sin(n\,x). \tag{3.122}$$

On the other hand, at $t=0$ we have

$$\left(1 - \frac{1}{\pi}\right) x = v(x,0) = \sum_{n=1}^{\infty} T_n(0) \sin(n\,x); \tag{3.123}$$

and

$$1 = \frac{\partial v}{\partial t}(x,0) = \sum_{n=1}^{\infty} T_n'(0) \sin(n\,x). \tag{3.124}$$

Therefore, if we expand $x\,t$, $\left(1-\frac{1}{\pi}\right)x$, 1 into their Fourier Sine series:

$$x\,t = \sum_{n=1}^{\infty} h_n(t) \sin(n\,x); \tag{3.125}$$

$$\left(1 - \frac{1}{\pi}\right) x = \sum_{n=1}^{\infty} a_n \sin(n\,x); \tag{3.126}$$

$$1 = \sum_{n=1}^{\infty} b_n \sin(n\,x); \tag{3.127}$$

then T_n satisfies the initial value problem

$$T_n'' + 4\,n^2\,T_n = h_n(t); \qquad T_n(0) = a_n, \quad T_n'(0) = b_n. \tag{3.128}$$

Now we compute:

- $h_n(t)$:

$$\begin{aligned} h_n(t) &= \frac{2}{\pi} \int_0^{\pi} (x\,t) \sin(n\,x)\,\mathrm{d}x \\ &= \frac{2\,t}{\pi} \int_0^{\pi} x \sin(n\,x)\,\mathrm{d}x \\ &= \frac{2\,t}{\pi} \left(-\frac{1}{n}\right) \int_0^{\pi} x\,\mathrm{d}\cos(n\,x) \\ &= -\frac{2\,t}{n\,\pi} \left[x \cos(n\,x)|_0^{\pi} - \int_0^{\pi} \cos(n\,x)\,\mathrm{d}x \right] \\ &= -\frac{2\,t}{n\,\pi} [\pi\,(-1)^n - 0] \\ &= (-1)^{n+1} \frac{2}{n}\,t. \end{aligned} \tag{3.129}$$

- a_n:

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{1}{\pi}\right) x \sin(n\,x)\,\mathrm{d}x \\ &= \left(1 - \frac{1}{\pi}\right) \left[\frac{2}{\pi} \int_0^{\pi} x \sin(n\,x)\,\mathrm{d}x \right] \\ &= \left(1 - \frac{1}{\pi}\right) (-1)^{n+1} \frac{2}{n} \\ &= (-1)^{n+1} \frac{2\,(\pi - 1)}{n\,\pi}. \end{aligned} \tag{3.130}$$

Note that we have taken advantage of the fact that $a_n = \left(1 - \frac{1}{\pi}\right) h_n(t)/t$.

- b_n:

$$b_n = \frac{2}{\pi} \int_0^\pi \sin(n\,x)\,\mathrm{d}x = -\frac{2}{n\,\pi}[(-1)^n - 1] = \frac{2}{n\,\pi}[(-1)^{n+1} + 1]. \tag{3.131}$$

Summarizing, T_n satisfies

$$T_n'' + 4\,n^2\,T_n = (-1)^{n+1}\frac{2}{n}t;\ T_n(0) = (-1)^{n+1}\frac{2\,(\pi - 1)}{n\,\pi};\ T_n'(0) = \frac{2}{n\,\pi}[(-1)^{n+1} + 1]. \tag{3.132}$$

- Solve T_n.

As T_n satisfies "nonhomogeneous 2nd order linear constant coefficient equation", we have to solve the corresponding homogeneous equation

$$T'' + 4\,n^2\,T = 0 \tag{3.133}$$

first to get its general solution, and then find a particular solution of $T_n'' + 4\,n^2\,T_n = (-1)^{n+1}\frac{2}{n}t$. Inspecting the right hand side, we conclude that the best approach to get the particular solution should be "undetermined coefficients".

  - $T'' + 4\,n^2\,T = 0$. The general solution is

$$T = C_1 \cos(2\,n\,t) + C_2 \sin(2\,n\,t). \tag{3.134}$$

  - Particular solution. The correct form is

$$T_p = A\,t. \tag{3.135}$$

  Substitute into the equation we easily obtain

$$T_p = \frac{(-1)^{n+1}}{2\,n^3}t. \tag{3.136}$$

Thus T_n has to be of the form

$$T_n = C_1 \cos(2\,n\,t) + C_2 \sin(2\,n\,t) + \frac{(-1)^{n+1}}{2\,n^3}t. \tag{3.137}$$

Now enforce the initial conditions:

$$T_n(0) = (-1)^{n+1}\frac{2\,(\pi - 1)}{n\,\pi} \Longrightarrow C_1 = (-1)^{n+1}\frac{2\,(\pi - 1)}{n\,\pi}; \tag{3.138}$$

$$T_n'(0) = \frac{2}{n\,\pi}[(-1)^{n+1} + 1] \Longrightarrow 2\,n\,C_2 + \frac{(-1)^{n+1}}{2\,n^3} = \frac{2}{n\,\pi}[(-1)^{n+1} + 1] \tag{3.139}$$

which gives

$$C_1 = (-1)^{n+1}\frac{2\,(\pi - 1)}{n\,\pi}, \qquad C_2 = \frac{1}{n^2\,\pi}[(-1)^{n+1} + 1] - \frac{(-1)^{n+1}}{4\,n^4}. \tag{3.140}$$

Therefore

$$T_n(t) = (-1)^{n+1}\frac{2\,(\pi - 1)}{n\,\pi}\cos(2\,n\,t) + \left(\frac{1}{n^2\,\pi}[(-1)^{n+1} + 1] - \frac{(-1)^{n+1}}{4\,n^4}\right)\sin(2\,n\,t) + \frac{(-1)^{n+1}}{2\,n^3}t. \tag{3.141}$$

- Write down the solution. We have

$$v(x,\,t) = \sum_{n=1}^{\infty}\left[(-1)^{n+1}\frac{2\,(\pi - 1)}{n\,\pi}\cos(2\,n\,t) + \left(\frac{1}{n^2\,\pi}[(-1)^{n+1} + 1] - \frac{(-1)^{n+1}}{4\,n^4}\right)\sin(2\,n\,t) + \frac{(-1)^{n+1}}{2\,n^3}t\right]\sin(n\,x), \tag{3.142}$$

and

$$u(x,\ t)\ =\ v(x,\ t)\ +\ w(x)\ =\ \frac{x}{\pi}\ +\ \sum_{n=1}^{\infty}\ \left[(-1)^{n+1}\ \frac{2\,(\pi-1)}{n\,\pi}\ \cos\ (2\ n\ t)\ +\ \left(\frac{1}{n^2\,\pi}\ [(-1)^{n+1}\ +\ 1]\ -\ \frac{(-1)^{n+1}}{4\,n^4}\right)\sin\,(2\,n\,t)+\frac{(-1)^{n+1}}{2\,n^3}\,t\right]\sin\,(n\,x). \tag{3.143}$$

3.0.3. Discussions.

The success of the above method relies on the following:

1. The process of separating the variables leads to a certain eigenvalue problem;
2. This eigenvalue problem yields a sequence of eigenvalues, and each eigenvalue has a one-dimensional eigenspace – that is any two eigenfunctions of the same eigenvalue are linearly dependent;
3. It is possible to expand any reasonably smooth function into a (infinite) linear combination of these eigenfunctions.

The first is easy to verify. The second and third though are not obvious at all. When the eigenfunctions are $\sin\frac{n\,\pi\,x}{L}$ or $\cos\frac{n\,\pi\,x}{L}$, we have seen in earlier PDE courses that 2. can be shown by directly solving the eigenvalue problem, while 3. follows from the theory of Fourier sine/cosine series. In the general case, we need the so-called Sturm-Liouville theory, which will be discussed in sections 3.2 and 3.3.

Exercises.

Exercise 3.1. Consider the Telegrapher's equation

$$u_{x\,x}=u_{t\,t}+\lambda\,u_t \tag{3.144}$$

(recall that $\lambda>0$) over the interval $x\in[0,L]$ subject to conditions

$$u(0,t)=u(L,t)=0;\qquad u(x,0)=f(x),\quad u_t(x,0)=h(x). \tag{3.145}$$

Use the method of separation of variables to study the limiting behavior of u as $t\longrightarrow\infty$.

Exercise 3.2. Consider the heat equation with Dirichlet boundary condition:

$$u_t=\kappa\,u_{x\,x},\qquad u(0,t)=u(\pi,t)=0. \tag{3.146}$$

Now consider the semi-discretization of the equation: Replace $u_{x\,x}$ by $\frac{u(x+h,\,t)\,-\,2\,u(x,\,t)\,+\,u(x\,-\,h,\,t)}{h^2}$. If we set $U_i(t)=u(i\,h,\,t)$ where $h=L/n$, the equation becomes an ODE system:

$$U_t=A\,U,\qquad U_0(t)=U_n(t)=0 \tag{3.147}$$

with $U=\begin{pmatrix}U_0(t)\\ \vdots\\ U_n(t)\end{pmatrix}$, and $A=\frac{\kappa}{h^2}\begin{pmatrix}-2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2\end{pmatrix}$ $\left(a_{i\,j}=\begin{cases}-2 & i=j\\ 1 & |i-j|=1\\ 0 & |i-j|>1\end{cases}\right)$.

a) Show that $U^{(m)}$ with $U_j^{(m)}=\sin\left(m\,\frac{j\,\pi}{n}\right)$ is an eigenvector of the matrix A.

b) Show that any solution to (3.147) can be obtained as follows: Set $U=\sum v_m(t)\,U^{(m)}$ and solve an ODE for $v_m(t)$.

c) By taking limit $n\longrightarrow\infty$, formally justify the method of separation of variables for (3.146).

Exercise 3.3. Consider the arbitrary linear first order PDE:

$$a(x,y)\,u_x+b(x,y)\,u_y+c(x,y)\,u=d(x,y). \tag{3.148}$$

For what a — d is this equation solvable through separation of variables? Describe the solution procedure and use it to solve the equation

$$x\,u_x+y\,u_y=u. \tag{3.149}$$

Exercise 3.4. Consider an arbitrary linear second order PDE:

$$a(x,y)\,u_{x\,x}+b(x,y)\,u_{x\,y}+c(x,y)\,u_{y\,y}+d(x,y)\,u_x+e(x,y)\,u_y+f(x,y)\,u=g(x,y). \tag{3.150}$$

For what a — g is this equation solvable through separation of variables?

3.1. Higher Dimensional Problems and Special Functions.

3.1.1. Rectangular domains.

Example 3.7. Find a formal solution to the initial-boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad 0<x<\pi,\ 0<y<\pi,\ t>0 && (3.151)\\ \frac{\partial u}{\partial x}(0,y,t) &= \frac{\partial u}{\partial x}(\pi,y,t)=0, \qquad 0<y<\pi,\ t>0 && (3.152)\\ u(x,0,t) &= u(x,\pi,t)=0, \qquad 0<x<\pi,\ t>0 && (3.153)\\ u(x,y,0) &= y, \qquad 0<x<\pi,\ 0<y<\pi. && (3.154)\end{aligned}$$

Solution. We follow the procedure of separation of variables.

1. Separate the variables. Write $u(x,y,t)=X(x)\,Y(y)\,T(t)$. Substitute into the equation, we have

$$T'XY = TX''Y + TXY''. \tag{3.155}$$

Divide both sides by TXY:

$$\frac{T'}{T} = \frac{X''}{X} + \frac{Y''}{Y}. \tag{3.156}$$

As the left hand side only depends on t while the right hand side is independent of t, both sides have to be constant. Applying the same argument one more time, we conclude that

$$\frac{X''}{X} = \lambda, \qquad \frac{Y''}{Y} = \mu, \qquad \frac{T'}{T} = \lambda+\mu. \tag{3.157}$$

The equations for X, Y, T are then

$$X'' - \lambda X = 0, \qquad X'(0) = X'(\pi) = 0; \tag{3.158}$$

$$Y'' - \mu Y = 0, \qquad Y(0) = Y(\pi) = 0; \tag{3.159}$$

$$T' - (\lambda+\mu)\,T = 0. \tag{3.160}$$

2. Solve the eigenvalue problems. Now there are two eigenvalue problems. We solve them one by one.

   i. Solve X_n.

   $$X'' - \lambda X = 0, \qquad X'(0) = X'(\pi) = 0; \tag{3.161}$$

   We have eigenvalues $\lambda_n = -n^2$, $n=0,1,2,\ldots$ and eigenfunctions $X_n = a_n \cos(n\,x)$, $n=1,2,3$.

   ii. Solve Y_m.

   $$Y'' - \mu Y = 0, \qquad Y(0) = Y(\pi) = 0; \tag{3.162}$$

   We have eigenvalues $\mu_m = -m^2$, $m=1,2,3,\ldots$ and eigenfunctions $Y_m = b_m \sin(m\,y)$, $m=1,2,3,\ldots$

3. Solve $T_{n,m}$. We have

$$T'_{n,m} + (n^2+m^2)\,T_{n,m} = 0 \Longrightarrow T_{n,m} = T_{n,m}(0)\,e^{-(n^2+m^2)t}. \tag{3.163}$$

4. Write

$$u(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{nm}\cos(n\,x)\sin(m\,y)\,e^{-(n^2+m^2)t}. \tag{3.164}$$

5. Compute the coefficients. We have

$$u(x,y,0) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{nm}\cos(n\,x)\sin(m\,y)\,e^{-(n^2+m^2)t}. \tag{3.165}$$

To determine the coefficients, we first need to understand the integrations

$$\int_0^\pi \int_0^\pi \cos(n\,x)\sin(m\,y)\cos(n'\,x)\sin(m'\,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.166}$$

We compute

$$\begin{aligned}\int_0^\pi \int_0^\pi \cos(n\,x)\sin(m\,y)\cos(n'\,x)\sin(m'\,y)\,\mathrm{d}x\,\mathrm{d}y \;&=\; \left[\int_0^\pi \cos(n\,x)\cos(n'\,x)\,\mathrm{d}x\right]\cdot\\ &\quad\cdot\left[\int_0^\pi \sin(m\,y)\sin(m'\,y)\,\mathrm{d}y\right].\end{aligned} \tag{3.167}$$

Recall that for $n, n' \in \{0, 1, 2, 3, \ldots\}$ we have

$$\int_0^\pi \cos(n\,x)\cos(n'\,x)\,\mathrm{d}x = \begin{cases} \pi & n = n' = 0\\ \frac{\pi}{2} & n = n' \neq 0\\ 0 & n \neq n' \end{cases}, \tag{3.168}$$

and for $m, m' \in \{1, 2, 3, \ldots\}$

$$\int_0^\pi \sin(m\,y)\sin(m'\,y)\,\mathrm{d}y = \begin{cases} \frac{\pi}{2} & m = m'\\ 0 & m \neq m' \end{cases} \tag{3.169}$$

Therefore

$$\int_0^\pi \int_0^\pi \cos(n\,x)\sin(m\,y)\cos(n'\,x)\sin(m'\,y)\,\mathrm{d}x\,\mathrm{d}y = \begin{cases} \frac{\pi^2}{2} & n = n' = 0, m = m'\\ \frac{\pi^2}{4} & n = n' \neq 0, m = m'\\ 0 & n \neq n' \text{ or } m \neq m' \end{cases}. \tag{3.170}$$

As a consequence,

$$\int_0^\pi \int_0^\pi u(x, y, 0)\cos(n\,x)\sin(m\,y)\,\mathrm{d}x\,\mathrm{d}y = \begin{cases} \frac{\pi^2}{2}\,c_{0m} & n = 0\\ \frac{\pi^2}{4}\,c_{nm} & n = 1, 2, 3, \ldots \end{cases}. \tag{3.171}$$

Now we compute

$$\begin{aligned}\int_0^\pi \int_0^\pi y\sin(m\,y)\,\mathrm{d}x\,\mathrm{d}y \;&=\; \pi \int_0^\pi y\sin(m\,y)\,\mathrm{d}y\\ &=\; -\frac{\pi}{m}\int_0^\pi y\,\mathrm{d}\cos(m\,y)\\ &=\; -\frac{\pi}{m}\left[y\cos(m\,y)|_0^\pi - \int_0^\pi \cos(m\,y)\,\mathrm{d}y\right]\\ &=\; -\frac{\pi^2}{m}(-1)^m.\end{aligned} \tag{3.172}$$

Thus

$$c_{0m} = \frac{2}{m}(-1)^{m+1}. \tag{3.173}$$

$$\int_0^\pi \int_0^\pi y\cos(n\,x)\sin(m\,y)\,\mathrm{d}x\,\mathrm{d}y \;=\; \left[\int_0^\pi \cos(n\,x)\,\mathrm{d}x\right]\left[\int_0^\pi y\sin(m\,y)\,\mathrm{d}y\right] = 0. \tag{3.174}$$

Therefore

$$c_{nm} = 0, \qquad n = 1, 2, 3, \ldots; m = 1, 2, 3, \ldots \tag{3.175}$$

Summarizing, we have

$$u(x, y, t) = \frac{2}{m}\sum_{m=1}^\infty (-1)^{m+1}\,e^{-m^2 t}\sin(m\,y). \tag{3.176}$$

3.1.2. Laplace's equations in polar and spherical coordinates.

We consider Laplace's equation in the unit disc and unit ball.

Example 3.8. (Polar) Solve

$$u_{xx}+u_{yy}=0,\quad x^2+y^2<1;\qquad u=f(\theta)\quad x^2+y^2=1. \tag{3.177}$$

Solution. We get

$$\frac{r^2 R''}{R}+\frac{r R'}{R}=-\frac{\Theta''}{\Theta}=\lambda. \tag{3.178}$$

This leads to

$$\Theta''+\lambda\,\Theta=0;\qquad \Theta(0)=\Theta(2\,\pi),\quad \Theta'(0)=\Theta'(2\,\pi) \tag{3.179}$$

and

$$r^2 R''+r\,R'-\lambda\,R=0;\qquad R(0)\text{ bounded}; R(1)=f(\theta). \tag{3.180}$$

The Θ equation is easily solved, giving

$$\lambda_n=n^2,\qquad n=0,1,2,3,\ldots \tag{3.181}$$

$$\Theta_0=1,\qquad \Theta_{n1}=\cos n\,\theta,\ \Theta_{n2}=b_n\sin n\,\theta. \tag{3.182}$$

Substituting the λ_n's into the R equation and expanding $f(\theta)$ into Fourier series

$$f(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\,[a_n\cos n\,\theta+b_n\sin n\,\theta], \tag{3.183}$$

we obtain updated R equation as

$$r^2 R_0''+r\,R_0'=0,\qquad R_0(0)\text{ bounded},\quad R_0(1)=\frac{a_0}{2}; \tag{3.184}$$

$$r^2 R_{ni}''+r\,R_{ni}'-n^2 R=0,\qquad R_{ni}(0)\text{ bounded},\quad R_{ni}(1)=\begin{cases} a_n & i=1\\ b_n & i=2\end{cases}. \tag{3.185}$$

Notice that the R equation is Cauchy-Euler, which means it can be solved by setting $R=r^\alpha$ with α satisfying

$$\alpha\,(\alpha-1)+\alpha-n^2=0\Longrightarrow\alpha_{1,2}=\pm n. \tag{3.186}$$

When $n=0$, we have a repeated root, therefore the general solution for R_0 is

$$R_0(r)=C_1+C_2\ln r. \tag{3.187}$$

The boundary conditions then dictates that $C_2=0$, $C_1=\frac{a_0}{2}$; Similarly, we have $R_{n1}(r)=a_n\,r^n$ and $R_{n2}(r)=b_n\,r^n$.

Summarizing, the solution to the problem is

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\,r^n\,[a_n\cos n\,\theta+b_n\sin n\,\theta] \tag{3.188}$$

where a_n,b_n comes from (3.183), the Fourier expansion of f.

Example 3.9. (Spherical with rotational symmetry) We consider the Laplace equation

$$u_{xx}+u_{yy}+u_{zz}=0,\qquad u=f\text{ on }x^2+y^2+z^2=1. \tag{3.189}$$

It is clear that we should turn to spherical coordiantes

$$x=r\cos\theta\sin\varphi,\qquad y=r\cos\theta\cos\varphi,\qquad z=r\cos\varphi \tag{3.190}$$

where θ is the angle (on x-y plane) from the x axis (that is $\tan\theta=y/x$), and φ the vertical angle from the z axis (that is $\cos\varphi=z/r$). Clearly $0<\theta\leqslant 2\,\pi$, $0\leqslant\varphi\leqslant\pi$.

The equation now becomes

$$u_{rr}+\frac{2}{r}\,u_r+\frac{1}{r^2}\left[u_{\varphi\varphi}+(\cot\varphi)\,u_\varphi+\frac{1}{\sin^2\varphi}\,u_{\theta\theta}\right]=0,\qquad u(1,\theta,\varphi)=f(\theta,\varphi). \tag{3.191}$$

We first consider the case where f has rotational symmetry, that is $f = f(\varphi)$. Then it is reasonable to expect $u = u(r, \varphi)$.

The problem now reduces to

$$
\begin{aligned}
u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} [u_{\varphi\varphi} + (\cot \varphi) u_\varphi] &= 0, \qquad 0 < r < 1, 0 < \varphi < \pi && (3.192)\\
u(1, \varphi) &= f(\varphi), \qquad 0 < \varphi < \pi && (3.193)
\end{aligned}
$$

Let $u(r, \varphi) = R(r) \Phi(\varphi)$, we reach

$$r^2 R'' + 2 r R' - \lambda R = 0, \qquad R, R' \text{ bounded as } r \longrightarrow 0+; \tag{3.194}$$

$$\Phi'' + \cot \varphi \Phi' + \lambda \Phi = 0. \tag{3.195}$$

The Φ equation, under the change of variable $x = \cos \varphi$, becomes

$$(1 - x^2) y'' - 2 x y' + \lambda y = 0, \qquad -1 < x < 1. \tag{3.196}$$

where $y(x) := \Phi(\varphi)$.

The reasonable boundary condition for the y equations should be y, y' remain bounded as $x \longrightarrow 1-$ and $x \longrightarrow -1+$.

It turns out that such boundary condition already determines a list of eigenvalues and eigenfunctions:

$$\lambda_n = n (n + 1), \qquad n = 0, 1, 2, \ldots; \qquad P_n = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} [(x^2 - 1)^n]. \tag{3.197}$$

These P_n's are called Legendre polynomials.

With λ_n's known, we can easily solve $R_n = r^n$. So the solution is

$$u(r, \varphi) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos \varphi) \tag{3.198}$$

where a_n comes from the expansion

$$f(\varphi) = \sum_{n=0}^{\infty} a_n P_n(\cos \varphi). \tag{3.199}$$

with

$$a_n = \frac{\int_0^\pi f(\varphi) P_n(\cos \varphi) \sin \varphi \, \mathrm{d}\varphi}{\int_0^\pi P_n^2(\cos\varphi) \sin\varphi \, \mathrm{d}\varphi}. \tag{3.200}$$

Example 3.10. (Spherical, general case) Now we consider the general case

$$u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} \left[u_{\varphi\varphi} + (\cot \varphi) u_\varphi + \frac{1}{\sin^2\varphi} u_{\theta\theta} \right] = 0, \qquad u(1, \theta, \varphi) = f(\theta, \varphi). \tag{3.201}$$

Setting $u(r, \theta, \varphi) = R(r) \Theta(\theta) \Phi(\varphi)$, we reach

$$
\begin{aligned}
r^2 R'' + 2 r R' - \lambda R &= 0, && (3.202)\\
\Theta'' + \mu \Theta &= 0, && (3.203)\\
\Phi'' + \cot \varphi \Phi' + \left(\lambda - \mu \frac{1}{\sin^2\varphi} \right) \Phi &= 0. && (3.204)
\end{aligned}
$$

Here R, Φ subject to similar boundary conditions as in the last example, while Θ enjoys the periodic boundary condition.

It is clear that Θ should be solved first to yield

$$\mu_m = m^2, \qquad m = 0, 1, 2, \ldots \tag{3.205}$$

with eigenfunctions 1 (for $m = 0$) and $\cos m \theta, \sin m \theta$ for $m = 1, 2, 3, \ldots$.

Taking the change of variable $x = \cos \varphi$ and set $y(x) := \Phi(\varphi)$ we reach

$$(1 - x^2) y'' - 2 x y' + \left(\lambda - \frac{m^2}{1 - x^2} \right) y = 0, \qquad -1 < x < 1. \tag{3.206}$$

This is called *associated Legendre's equation*. The eigenvalues are still $n\,(n+1)$, $n=0,1,2,\ldots$ while the (bounded) eigenfunctions are now the *associated Legendre functions of first kind*

$$P_n^m(x) := (-1)^m\,(1-x^2)^{m/2}\,\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_n(x) \tag{3.207}$$

where $P_n(x)$ are the Legendre polynomials in the last example. Note that the above formula gives $P_n^m(x)=0$ when $m>n$.

The solution now reads

$$u(\rho,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} r^n\,[a_{n\,m}\cos(m\,\theta)+b_{n\,m}\sin(m\,\theta)]\,P_n^m(\cos\varphi) \tag{3.208}$$

with $a_{n\,m}, b_{n\,m}$ given by

$$a_{n\,m} = \frac{\int_0^\pi\int_0^{2\pi} f(\theta,\varphi)\,P_n^m(\cos\varphi)\cos(m\,\theta)\sin\varphi\,\mathrm{d}\theta\mathrm{d}\varphi}{\int_0^\pi\int_0^{2\pi}[P_n^m(\cos\varphi)\cos(m\,\theta)]^2\sin\varphi\,\mathrm{d}\theta\,\mathrm{d}\varphi}; \tag{3.209}$$

$$b_{n\,m} = \frac{\int_0^\pi\int_0^{2\pi} f(\theta,\varphi)\,P_n^m(\cos\varphi)\sin(m\,\theta)\sin\varphi\,\mathrm{d}\theta\mathrm{d}\varphi}{\int_0^\pi\int_0^{2\pi}[P_n^m(\cos\varphi)\sin(m\,\theta)]^2\sin\varphi\,\mathrm{d}\theta\,\mathrm{d}\varphi}. \tag{3.210}$$

Remark 3.11. Note that we can expand any function defined on the sphere by $\cos(m\,\theta)\,P_n^m(\cos\varphi)$ and $\sin(m\,\theta)\,P_n^m(\sin\varphi)$. This is called "spherical harmonics" expansion.

3.1.3. Heat equation in the cylinder.

Example 3.12. Consider the heat equation in a 2D disc $x^2+y^2\leqslant 1$:

$$\begin{aligned} u_t &= \kappa\,(u_{x\,x}+u_{y\,y}) &\quad (3.211)\\ u(x,y,0) &= f(x,y) &\quad (3.212)\\ u(x,y,t) &= 0 \qquad\qquad x^2+y^2=1. &\quad (3.213)\end{aligned}$$

Solution. Due to the special geometry of the domain, it is natural to consider the problem using polar coordinates (r,θ) satisfying

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{3.214}$$

Now we change the variables from x,y to r,θ. Differentiating the above relation we have

$$\begin{aligned} (\cos\theta)\,r_x - r\,(\sin\theta)\,\theta_x &= 1 &\quad (3.215)\\ (\cos\theta)\,r_y - r\,(\sin\theta)\,\theta_y &= 0 &\quad (3.216)\\ (\sin\theta)\,r_x + r\,(\cos\theta)\,\theta_x &= 0 &\quad (3.217)\\ (\sin\theta)\,r_y + r\,(\cos\theta)\,\theta_y &= 1 &\quad (3.218)\end{aligned}$$

consequently

$$r_x = \frac{x}{r}, \quad r_y = \frac{y}{r}, \quad r_{x\,x} = \frac{1}{r}-\frac{x^2}{r^3}, \quad r_{y\,y} = \frac{1}{r}-\frac{y^2}{r^3}; \tag{3.219}$$

$$\theta_x = -\frac{\sin\theta}{r} = -\frac{y}{r^2}, \quad \theta_y = \frac{\cos\theta}{r} = \frac{x}{r^2}, \quad \theta_{x\,x} = \frac{2\,x\,y}{r^4}, \quad \theta_{y\,y} = -\frac{2\,x\,y}{r^4}$$

Therefore

$$\begin{aligned} u_{x\,x} &= u_{r\,r}\frac{x^2}{r^2} - u_{r\theta}\frac{2\,x\,y}{r^3} + u_{\theta\theta}\frac{y^2}{r^4} + u_r\left(\frac{1}{r}-\frac{x^2}{r^3}\right) + u_\theta\,\frac{2\,x\,y}{r^4}, &\quad (3.220)\\ u_{y\,y} &= u_{r\,r}\frac{y^2}{r^2} + u_{r\theta}\frac{2\,x\,y}{r^3} + u_{\theta\theta}\frac{x^2}{y^4} + u_r\left(\frac{1}{r}-\frac{y^2}{r^3}\right) + u_\theta\left(-\frac{2\,x\,y}{r^4}\right). &\quad (3.221)\end{aligned}$$

The equation and the initial-boundary conditions in polar coordinate form are

$$\begin{aligned} u_t &= \kappa\left(u_{r\,r}+\frac{1}{r}\,u_r+\frac{1}{r^2}\,u_{\theta\theta}\right) &\quad (3.222)\\ u(r,\theta,0) &= f(r,\theta) &\quad (3.223)\\ u(1,\theta,t) &= 0. &\quad (3.224)\end{aligned}$$

We apply separation of variables to solve this equation.

First we try to find non-trivial "basic" solutions of the form

$$u(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t). \tag{3.225}$$

Substituting this into the equation we reach

$$R(r)\,\Theta(\theta)\,T'(t) = \kappa\left(R''(r)\,\Theta(\theta) + \frac{1}{r}\,R'(r)\,\Theta(\theta) + \frac{1}{r^2}\,R(r)\,\Theta''(\theta)\right)T(t). \tag{3.226}$$

Dividing both sides by $R(r)\,\Theta(\theta)\,T(t)$ we reach

$$\frac{T'(t)}{T(t)} = \kappa\left(\frac{R''(r)}{R(r)} + \frac{1}{r}\,\frac{R'(r)}{R(r)} + \frac{1}{r^2}\,\frac{\Theta''(\theta)}{\Theta(\theta)}\right). \tag{3.227}$$

As the LHS only involves t and the RHS only r, θ there is a constant λ such that

$$\frac{T'(t)}{T(t)} = -\kappa\,\lambda \tag{3.228}$$

and

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\,\frac{R'(r)}{R(r)} + \frac{1}{r^2}\,\frac{\Theta''(\theta)}{\Theta(\theta)} = -\lambda. \tag{3.229}$$

Multiply both sides by r^2 we have

$$\frac{r^2\,R''(r)}{R(r)} + \frac{r\,R'(r)}{R(r)} + \lambda\,r^2 = \frac{\Theta''(\theta)}{\Theta(\theta)}. \tag{3.230}$$

The LHS only involves r and the RHS only θ. Thus there is a constant μ such that

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = \mu, \qquad \frac{r^2\,R''(r)}{R(r)} + \frac{r\,R'(r)}{R(r)} + \lambda\,r^2 = -\mu. \tag{3.231}$$

As $\Theta(\theta)$ is obviously $2\,\pi$ periodic, we have

$$\mu = -n^2, \qquad n = 1, 2, 3, \ldots \tag{3.232}$$

and

$$\Theta(\theta) = A\cos(n\,\theta) + B\sin(n\,\theta). \tag{3.233}$$

On the other hand, the equation for R now becomes

$$r^2\,R'' + r\,R' + (\lambda\,r^2 - n^2)\,R = 0, \tag{3.234}$$

with the boundary condition

$$R(1) = 0, \qquad R(0) \text{ bounded}. \tag{3.235}$$

The general solution is

$$R(r) = C_1\,J_n\big(\sqrt{\lambda}\,r\big) + C_2\,Y_n\big(\sqrt{\lambda}\,r\big) \tag{3.236}$$

with J_n, Y_n Bessel functions of the first and second kinds. It turns out that $Y_n(r)$ is unbounded as $r \longrightarrow 0+$, therefore we have

$$R_n(r) = J_n\big(\sqrt{\lambda}\,r\big). \tag{3.237}$$

Applying the boundary condition $R(1) = 0$ gives

$$R_{n,k}(r) = J_n(\alpha_{n,k}\,r), \qquad \lambda_{n,k} = \alpha_{n,k}^2. \tag{3.238}$$

Now we expand

$$f(r,\theta) = \sum_{n,k} a_{n,k}\,R_{n,k}(r)\cos(n\,\theta) + b_{n,k}\,R_{n,k}\sin(n\,\theta). \tag{3.239}$$

and the solution is given by

$$\sum_{n,k}\,[a_{n,k}\,R_{n,k}(r)\cos(n\,\theta) + b_{n,k}\,R_{n,k}\sin(n\,\theta)]\,e^{-\lambda_{n,k}t} \tag{3.240}$$

3.1.4. Discussions.

We see that similar to the 1D situation, the success of the above method relies on the following:

1. The process of separating the variables leads to a certain eigenvalue problem;
2. This eigenvalue problem yields a sequence of eigenvalues, and each eigenvalue has a one-dimensional eigenspace – that is any two eigenfunctions of the same eigenvalue are linearly dependent;
3. It is possible to expand any reasonably smooth function into a (infinite) linear combination of these eigenfunctions.

In the higher dimensional case, it is quite unclear which weight we should use for the expansion formulas.

Reference.

- John M. Davis, "Introduction to Applied Partial Differential Equations", Chap. 5, 6.

Exercises.

Exercise 3.5. Consider the boundary value problem for $u(x, y)$ in the annular region:

$$u_{xx} + u_{yy} = 0 \quad \rho^2 < x^2 + y^2 < 1; \qquad u(x, y) = \begin{cases} f & x^2 + y^2 = \rho^2 \\ g & x^2 + y^2 = 1 \end{cases}. \tag{3.241}$$

Obtain the formula for the solution using separation of variables.

Exercise 3.6. Consider the Laplace equation in the disc $x^2 + y^2 < \rho^2$, with boundary condition $u = f$. Using separation of variables, derive Poisson's formula:

$$u(r, \theta) = \frac{\rho^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\varphi)}{\rho^2 - 2\rho r \cos(\theta - \varphi) + r^2} \, \mathrm{d}\varphi. \tag{3.242}$$

Can we obtain a similar formula for solutions of the above annulus problem?

Exercise 3.7. Using Poisson's formula (3.242) to prove the following mean value property:

$$u(0, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi) \, \mathrm{d}\varphi. \tag{3.243}$$

Then establish the following maximum principle:

If $u_{xx} + u_{yy} = 0$ for $(x, y) \in \Omega \subseteq \mathbb{R}^2$, and $u = f$ on $\partial\Omega$, then $\max_\Omega u = \max_{\partial\Omega} f$.

Can this line of argument be used to prove the uniqueness of the classical solution to $u_{xx} + u_{yy} = g$ in Ω, $u = f$ on $\partial\Omega$? Why?

Exercise 3.8. Consider the equation

$$r^2 R'' + r R' + (\lambda r^2 - n^2) R = 0, \qquad R(0) \text{ bounded}, \ R(1) = 0. \tag{3.244}$$

a) Prove that there are no negative eigenvalues.

b) Prove that $\lambda = 0$ is not an eigenvalue.

c) Let $\lambda_k \neq \lambda_l$ be eigenvalues, prove that the eigenfunctions $R_k(r), R_l(r)$ satisfy

$$\int_0^1 R_k(r) \, R_l(r) \, r \, \mathrm{d}r = 0. \tag{3.245}$$

Exercise 3.9. Let $n \in \mathbb{N}$. Consider the equation

$$(1 - x^2) \, y'' - 2 x y' + n(n+1) \, y = 0, \qquad -1 < x < 1 \tag{3.246}$$

Use power series method to show that the general solution is $y = c_1 P_n + c_2 Q_n$ where P_n is a polynomial of degree n, and Q_n is unbounded at $1-$ and $-1+$.

Exercise 3.10. Solve the vibrating drum:

$$u_{tt} = u_{xx} + u_{yy} = u_{rr} + r^{-1} u_r + r^{-2} u_{\theta\theta}. \tag{3.247}$$

with

$$u(r, \theta, 0) = f(r), \qquad u_t(r, \theta, 0) = 0, \qquad u(1, \theta, t) = 0. \tag{3.248}$$

(Ans: $\sum c_n J_0(k_n r) \cos(k_n t)$).

3.2. Sturm-Liouville theory.

First recall how we prove convergence for the 1D cases. This relies on the explicit formula for partial sums and cannot be easily generalized.

3.2.1. Sturm-Liouville problems.

The standard Sturm-Liouville (SL) problem is of the form

$$(p(x)\,y')' + q(x)\,y + \lambda\,r(x)\,y = 0, \qquad a < x < b \tag{3.249}$$

$$\alpha_0\,y(a) + \alpha_1\,y'(a) = 0, \tag{3.250}$$

$$\beta_0\,y(b) + \beta_1\,y'(b) = 0. \tag{3.251}$$

where all the functions and numbers are real. For simplicity we assume the coefficients are as smooth as we need.

The problem is called

- *regular* when p, q, r are bounded on $[a, b]$ (that is the interval $a \leqslant x \leqslant b$), $p, r > 0$ for all $a \leqslant x \leqslant b$, and α_0, α_1 real, not both 0, and β_0, β_1 real, not both 0.
- *singular* when any one or more of the following happens
  - → The interval (a, b) is infinite, that is either $a = -\infty$ or $b = +\infty$ or both occurs.
  - → $p(x) = 0$ for some $x \in [a, b]$ or $r(x) = 0$ for some $x \in [a, b]$.
  - → One or several coefficient function becomes ∞ at a or b, or both.

Example 3.13. We check the systems we have dealt with

- $$y'' + \lambda\,y = 0, \qquad y(0) = y(l) = 0 \tag{3.252}$$

  We have

  $$a = 0,\ b = l;\ p(x) = 1,\ q(x) = 0,\ r(x) = 1;\ \alpha_0 = 1,\ \alpha_1 = 0,\ \beta_0 = 1,\ \beta_1 = 0. \tag{3.253}$$

  The system is a regular SL problem.

- $$y'' + \lambda\,y = 0, \qquad y'(0) = y'(l) = 0 \tag{3.254}$$

  We have

  $$a = 0,\ b = l;\ p(x) = 1,\ q(x) = 0,\ r(x) = 1;\ \alpha_0 = 0,\ \alpha_1 = 1,\ \beta_0 = 0,\ \beta_1 = 1. \tag{3.255}$$

  This is also a regular SL problem.

- $$y'' + \lambda\,y = 0, \qquad y(0) = 0,\ y'(l) = -h\,y(l). \tag{3.256}$$

  We have

  $$a = 0,\ b = l;\ p(x) = 1,\ q(x) = 0,\ r(x) = 1;\ \alpha_0 = 1,\ \alpha_1 = 0,\ \beta_0 = h,\ \beta_1 = 1. \tag{3.257}$$

- $$x^2\,y'' + x\,y' + (\lambda\,x^2 - n^2)\,y = 0, \qquad y(0)\text{ bounded},\ y(1) = 0. \tag{3.258}$$

  At first sight this problem is not an SL problem. However we can transform it through the following operations:

  We search for a multiplier $h(x)$ such that

  $$h(x)\,[x^2\,y'' + x\,y' + (\lambda\,x^2 - n^2)\,y] = (p\,y')' + q\,y + \lambda\,r\,y. \tag{3.259}$$

Comparing the two sides, we have

$$h(x)\,x^2 = p(x), \qquad h(x)\,x = p(x)' \tag{3.260}$$

which leads to

$$p(x)' = \frac{1}{x}\,p(x) \implies p(x) = x \implies h(x) = \frac{1}{x}. \tag{3.261}$$

Thus we see that the equation is equivalent to

$$(x\,y')' - \frac{n^2}{x}\,y + \lambda\,x\,y = 0 \tag{3.262}$$

which corresponds to

$$a = 0,\ b = 1;\ p(x) = x,\ q(x) = -\frac{n^2}{x},\ r(x) = x;\ \beta_0 = 1,\ \beta_1 = 0. \tag{3.263}$$

This is a singular SL problem.

Any λ that the problem has non-trivial solutions is called an eigenvalue, the corresponding solutions are called eigenfunctions.

3.2.2. Properties of regular Sturm-Liouville problems.

We see from the following theorem that the solutions to a SL problem enjoy similar properties as the functions $\sin\left(\frac{n\,\pi}{l}\,x\right)$ and $\cos\left(\frac{n\,\pi}{l}\,x\right)$ in the Fourier series. More specifically we have the following theorem.

Theorem 3.14. *A regular SL problem has the following properties.*

1. *About eigenvalues:*

   - *It has nonzero solutions for a countably infinite set of values of λ, called "eigenvalues" of the problem.*
   - *These eigenvalues are all real.*
   - *The set of eigenvalues does not have any limit points in $\mathbb{R}$.*
   - *These eigenvalues are bounded from below if $\alpha_0\,\alpha_1 \leqslant 0$ and $\beta_0\,\beta_1 \geqslant 0$. These eigenvalues are bounded from below by 0 if furthermore $q \leqslant 0$.*

   In summary, if we have $\alpha_0\ \alpha_1 \leqslant 0$ and $\beta_0\ \beta_1 \geqslant 0$, then then eigenvalues can be enumerated as: $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$.

2. *About eigenfunctions:*

   a) *For each fixed eigenvalue λ_n, the solution space is one-dimensional. That is, there is φ_n such that all other solutions for the same λ is a multiple of φ_n.*

   b) **(Orthogonality)** $\int_a^b \varphi_n(x)\,\varphi_m(x)\,r(x)\,\mathrm{d}x = 0$ *for any $n \neq m$.*

   c) **(Bessel's inequality)** *If φ_n's are chosen such that*

   $$\int_a^b \varphi_n(x)^2\,r(x)\,\mathrm{d}x = 1 \tag{3.264}$$

   We have the following Bessel's inequality

   $$\int_a^b f(x)^2\,r(x)\,\mathrm{d}x \geqslant \sum_{n=1}^{\infty} |c_n|^2. \tag{3.265}$$

d) **(Completeness)**

- *c1) The only continuous function f on $[a,b]$ with $\int_a^b f(x)\,\varphi_n(x)\,r(x)\,\mathrm{d}x=0$ for all n is $f\equiv 0$.*
- *c2) For any f having two continuous derivatives on $[a,b]$ and satisfying the boundary conditions, the infinite sum*
$$\sum_{n=1}^{\infty} c_n\,\varphi_n \tag{3.266}$$
where
$$c_n=\frac{\int_a^b f(x)\,\varphi_n(x)\,r(x)\,\mathrm{d}x}{\int_a^b \varphi_n(x)^2\,r(x)\,\mathrm{d}x} \tag{3.267}$$
converges absolutely uniformly to $f(x)$. By "absolutely uniformly" we mean
$$\sum_1^{\infty} |c_n|\,|\varphi_n|<\infty \tag{3.268}$$
and the convergence to f is uniform.
- *c3)* **(Parseval's equality)** *If a function $f(x)$ is continuous and φ_n's are chosen such that*
$$\int_a^b \varphi_n(x)^2\,r(x)\,\mathrm{d}x=1 \tag{3.269}$$
We have the following
$$\int_a^b f(x)^2\,r(x)\,\mathrm{d}x=\sum_{n=1}^{\infty} |c_n|^2. \tag{3.270}$$

Remark 3.15. We note that the "completeness" part is quite schizophrenia – $f(x)$ is continuous in one line but required to be twice continuously differentiable in the next. This will be resolved in later sections when we take a higher, functional analytic, point of view.

Example 3.16. We check the systems we have dealt with

–
$$y''+\lambda\,y=0, \qquad y(0)=y(l)=0 \tag{3.271}$$
We have $r(x)=1$, therefore the eigenfunctions satisfy
$$\int_0^l \varphi_n(x)\,\varphi_m(x)\,\mathrm{d}x=0 \qquad n\neq m \tag{3.272}$$
and the expansion reads
$$f(x)=\sum c_n\,\varphi_n, \qquad c_n=\frac{\int_0^l f(x)\,\varphi_n(x)\mathrm{d}x}{\int_0^l\,[\varphi_n(x)]^2\,\mathrm{d}x}. \tag{3.273}$$

–
$$y''+\lambda\,y=0, \qquad y'(0)=y'(l)=0 \tag{3.274}$$
We have again $r(x)=1$, thus although the eigenfunctions are now different, the orthogonality relation and expansion formula remain the same as above.

–

$$y'' + \lambda\, y = 0, \qquad y(0) = 0,\ \ y'(l) = -h\, y(l). \tag{3.275}$$

We have $r(x) = 1$. Thus the orthogonality relation and expansion formula remain the same – although we cannot write the explicit formulas for the eigenfunctions anymore!

– Bessel functions.

$$x^2\, y'' + x\, y' + (\lambda\, x^2 - n^2)\, y = 0, \qquad y(0) \text{ bounded},\ y(1) = 0. \tag{3.276}$$

Recall that by multiplication of $h(x) = 1/x$ the problem is turned into the Sturm-Liouville form

$$(x\, y')' - \frac{n^2}{x}\, y + \lambda\, x\, y = 0 \tag{3.277}$$

Although this problem is singular, it turns out that its eigenvalues/eigenfunctions enjoy similar properties as those in the regular case. In particuar, we have the orthogonality relation and expansion formulas:

$$\int_0^1 \varphi_n(x)\, \varphi_m(x)\, x\, \mathrm{d}x = 0, \qquad c_n = \frac{\int_0^1 f(x)\, \varphi_n(x)\, x\, \mathrm{d}x}{\int_0^1 [\varphi_n(x)]^2\, x\, \mathrm{d}x}. \tag{3.278}$$

– Legendre functions.

$$[(1 - x^2)\, y']' + \lambda\, y = 0, \qquad -1 < x < 1. \tag{3.279}$$

As $p(x) = 1 - x^2$ vanishes at both ends, the boundary conditions should be taken as

$$y, y' \text{ remain bounded as } x \rightarrow \pm 1. \tag{3.280}$$

The eigenvalues are $\lambda_n = n\,(n+1)$. Here $r(x) = 1$, so the corresponding eigenfunctions satisfy

$$\int_{-1}^1 P_m(x)\, P_n(x)\, \mathrm{d}x = 0, \qquad n \neq m. \tag{3.281}$$

– Hermite functions.

$$u'' - 2\, x\, u' + \lambda\, u = 0, \qquad -\infty < x < \infty \tag{3.282}$$

To write this problem into a SL problem, we multiply the equation by e^{-x^2} to obtain

$$\left(e^{-x^2} u'\right)' + \lambda\, e^{-x^2} u = 0, \qquad -\infty < x < \infty. \tag{3.283}$$

This is a regular S-L problem. Now that we have $p(x) = e^{-x^2}$ which tends to 0 as $x \rightarrow \pm\infty$, the boundary conditions should be

$$u, u' \text{ remain bounded as } x \rightarrow \pm\infty. \tag{3.284}$$

The eigenvalues are $\lambda_n = 2\, n$ for nonnegative integers n. Since $r(x) = e^{-x^2}$, the orthogonality property reads

$$\int_{-\infty}^{\infty} H_n(x)\, H_m(x)\, e^{-x^2}\, \mathrm{d}x = 0, \qquad n \neq m. \tag{3.285}$$

Proof. (Of the easy parts of theorem 3.14) The proofs for some of the above claims are more technical. We postpone them to the next lecture and only prove the easy ones (those in blue) here.

1. Properties of the eigenvalues.

    - These eigenvalues are all real.

Let λ be an eigenvalue and let φ be a corresponding eigenfunction. We compute

$$\begin{aligned} 0 &= \int_a^b [(p\,y')' + q\,y + \lambda\,r\,y]\,\bar{y}\,\mathrm{d}x \\ &= \int_a^b (p\,y')'\,\bar{y} + \int_a^b q\,|y|^2 + \lambda \int_a^b r\,|y|^2 \\ &= (p\,y')\,\bar{y}\,|_a^b - \int_a^b p\,y'\,\bar{y}\,' + \int_a^b q\,|y|^2 + \lambda \int_a^b r\,|y|. \end{aligned} \tag{3.286}$$

On the other hand, taking the complex conjugate of

$$(p\,y')' + q\,y + \lambda\,r\,y = 0 \tag{3.287}$$

we obtain

$$(p\,\bar{y}\,')' + q\,\bar{y} + \bar{\lambda}\,r\,\bar{y} = 0. \tag{3.288}$$

In other words, $\bar{\lambda}$ is also an eigenvalue with eigenfunction $\bar{y}$. Multiplying this equation by y and integrate, we have

$$\begin{aligned} 0 &= \int_a^b [(p\,\bar{y}\,')' + q\,\bar{y} + \bar{\lambda}\,r\,\bar{y}\,]\,y \\ &= \int_a^b (p\,\bar{y}\,')'\,y + \int_a^b q\,|y|^2 + \bar{\lambda} \int_a^b r\,|y|^2 \\ &= (p\,\bar{y}\,')\,y|_a^b - \int_a^b p\,y'\,\bar{y}\,' + \int_a^b q\,|y|^2 + \bar{\lambda} \int_a^b r\,|y|^2. \end{aligned} \tag{3.289}$$

Combining the above, we reach

$$(\lambda - \bar{\lambda}\,) \int_a^b r\,|y|^2 = p(b)\,[y'(b)\,\bar{y}\,(b) - \bar{y}\,'(b)\,y(b)] - p(a)\,[y'(a)\,\bar{y}\,(a) - \bar{y}\,'(a)\,y(a)]. \tag{3.290}$$

Using the boundary conditions

$$\alpha_0\,y(a) + \alpha_1\,y'(a) = 0, \tag{3.291}$$

$$\beta_0\,y(b) + \beta_1\,y'(b) = 0. \tag{3.292}$$

we see that

$$y'(b)\,\bar{y}\,(b) - \bar{y}\,'(b)\,y(b) = 0, \qquad y'(a)\,\bar{y}\,(a) - \bar{y}\,'(a)\,y(a) = 0. \tag{3.293}$$

Therefore

$$(\lambda - \bar{\lambda}\,) \int_a^b r\,|y|^2\,\mathrm{d}x = 0 \tag{3.294}$$

which leads to $\lambda = \bar{\lambda}$, or λ is real.

- These eigenvalues are bounded from below if $\alpha_0\,\alpha_1 \leqslant 0$ and $\beta_0\,\beta_1 \geqslant 0$. These eigenvalues are bounded from below by 0 if furthermore $q \leqslant 0$.

  We have

$$\begin{aligned} 0 &= \int_a^b [(p\,y')' + q\,y + \lambda\,r\,y]\,\bar{y}\,\mathrm{d}x \\ &= \int_a^b (p\,y')'\,\bar{y} + \int_a^b q\,|y|^2 + \lambda \int_a^b r\,|y|^2 \\ &= (p\,y')\,\bar{y}\,|_a^b - \int_a^b p\,y'\,\bar{y}\,' + \int_a^b q\,|y|^2 + \lambda \int_a^b r\,|y|^2. \\ &= p(b)\,y'(b)\,\bar{y}\,(b) - p(a)\,y'(a)\,\bar{y}\,(a) - \int_a^b [p\,|y'|^2 - q\,|y|^2] + \lambda \int_a^b r\,|y|^2. \end{aligned} \tag{3.295}$$

Thus

$$\lambda = \frac{\left\{-p(b)\, y'(b)\, \bar{y}\,(b) + p(a)\, y'(a)\, \bar{y}\,(a) + \int_a^b [p\,|y'|^2 - q\,|y|^2]\right\}}{\int_a^b r\,|y|^2}. \tag{3.296}$$

Using the boundary conditions we have

$$-p(b)\, y'(b)\, \bar{y}\,(b) = p(b)\, \frac{\beta_0}{\beta_1}\, |y(b)|^2, \tag{3.297}$$

$$p(a) y'(a)\, \bar{y}\,(a) = -p(a)\, \frac{\alpha_0}{\alpha_1}\, |y(a)|^2. \tag{3.298}$$

When $\alpha_0\, \alpha_1 \leqslant 0$ and $\beta_0\, \beta_1 \geqslant 0$, both terms are non-negative which means

$$\lambda \geqslant \frac{-\int_a^b q\,|y|^2}{\int_a^b r\,|y|^2}. \tag{3.299}$$

If furthermore $q \leqslant 0$, we see that $\lambda \geqslant 0$ too.

2. Properties of eigenfunctions.

- For each fixed eigenvalue λ, the solution space is one-dimensional. That is, there is y_λ such that all other solutions for the same λ is a multiple of y_λ.

  Fix λ. Let $y(x)$ and $z(x)$ be two eigenfunctions. That is

$$\begin{aligned} (p(x)\, y')' + q(x)\, y + \lambda\, r(x)\, y &= 0, \qquad a < x < b & (3.300)\\ \alpha_0\, y(a) + \alpha_1\, y'(a) &= 0, & (3.301)\\ \beta_0\, y(b) + \beta_1\, y'(b) &= 0. & (3.302) \end{aligned}$$

  and

$$\begin{aligned} (p(x)\, z')' + q(x)\, z + \lambda\, r(x)\, z &= 0, \qquad a < x < b & (3.303)\\ \alpha_0\, z(a) + \alpha_1\, z'(a) &= 0, & (3.304)\\ \beta_0\, z(b) + \beta_1\, z'(b) &= 0. & (3.305) \end{aligned}$$

  Multiplying the y equation by z and z equation by y, and subtract, we have

$$0 = (p\, y')'\, z - (p\, z')'\, y = (p\,(y'\, z - z'\, y))'. \tag{3.306}$$

  We conclude that

$$p(x)\,(y'\, z - z'\, y)(x) = p(a)\,(y'\, z - z'\, y)(a). \tag{3.307}$$

  As y, z both satisfy the boundary conditions, we have

$$p(a)\,(y'(a)\, z(a) - z'(a)\, y(a)) = 0 \tag{3.308}$$

  which leads to

$$p\,(y'\, z - z'\, y) = 0 \implies y'\, z - z'\, y = 0 \tag{3.309}$$

  for all $a \leqslant x \leqslant b$ as $p(x) > 0$.

  Finally,

$$y'\, z - z'\, y = 0 \implies \frac{y'}{y} = \frac{z'}{z} \implies \ln y - \ln z = \text{constant} \implies y/z = \text{constant}. \tag{3.310}$$

- **(Orthogonality)** $\int_a^b \varphi_n(x)\, \varphi_m(x)\, r(x)\, \mathrm{d}x = 0$ for any $n \neq m$.

  It suffices to show that if λ, μ are two distinct eigenvalues, and y, z the corresponding eigenfunctions, then $\int_a^b y\, z\, r\, \mathrm{d}x = 0$.

Using the equations we have

$$\int_a^b [(p\,y')' + q\,y + \lambda\,r\,y]\,z - [(p\,z')' + q\,z + \mu\,r\,z]\,y\,\mathrm{d}x = 0. \tag{3.311}$$

After using the boundary conditions, we can show that

$$\text{LHS} = (\lambda - \mu)\int y\,z\,r\,\mathrm{d}x. \tag{3.312}$$

Therefore

$$(\lambda - \mu)\int_a^b y(x)\,z(x)\,r(x)\,\mathrm{d}x = 0. \tag{3.313}$$

As $\lambda \neq \mu$, we have

$$\int_a^b y(x)\,z(x)\,r(x)\,\mathrm{d}x = 0. \tag{3.314}$$

- **(Bessel's inequality)** From orthogonality we have

$$0 \leqslant \int_a^b \left[f(x) - \sum_{n=1}^N c_n\,\varphi_n\right]^2 r(x)\mathrm{d}x = \int_a^b f(x)^2\,r(x)\,\mathrm{d}x - \sum_{n=1}^N |c_n|^2. \tag{3.315}$$

Taking limit $n \longrightarrow \infty$ we obtain Bessel's inequality. □

Exercises.

Exercise 3.11. Write the following equations into S-L form and discuss whether they are regular or singular. Determine what is the orthogonality relation their eigenfunctions should satisfy.

a) Legendre's equation:

$$(1 - x^2)\,y'' - 2\,x\,y' + \lambda\,y = 0, \qquad -1 < x < 1 \tag{3.316}$$

b) Chebyshev's equation

$$(1 - x^2)\,y'' - x\,y' + \lambda\,y = 0, \qquad -1 < x < 1 \tag{3.317}$$

c) Laguerre's equation

$$x\,y'' + (1 - x)\,y' + \lambda\,y = 0, \qquad 0 < x < \infty \tag{3.318}$$

d) Hermite's equation

$$y'' - 2\,x\,y' + \lambda\,y = 0, \qquad -\infty < x < \infty \tag{3.319}$$

e) Bessel's equation of order n

$$x^2\,y'' + x\,y' + (\lambda\,x^2 - n^2)\,y = 0, \qquad 0 < x < 1. \tag{3.320}$$

Exercise 3.12. Give any second order equation

$$a(x)\,y'' + b(x)\,y' + c\,(x)\,y = 0. \tag{3.321}$$

Prove that there exists a multiplier $h(x)$ such that

$$h(x)\,[a(x)\,y'' + b(x)\,y' + c\,(x)\,y] = (p(x)\,y')' + q(x)\,y. \tag{3.322}$$

Note that the term of first order derivative disappears.

Exercise 3.13. (Davis) Consider the S-L problem

$$(p\,y')' + q\,y + \lambda\,y = 0, \qquad a < x < b, \qquad y(a) = 0, y(b) = 0. \tag{3.323}$$

Show that if $p(x) \geqslant 0, q(x) \leqslant M$, then any eigenvalue $\lambda \geqslant -M$.

References.

- Anthony W. Knapp, "Advanced Real Analysis", §1.3.
- John M. Davis, "Introduction to Applied Partial Differential Equations", Chap. 4.

3.3. Proof of the Theorem (Difficult parts).

Recall that we are studying regular Sturm-Liouville problems:

$$
\begin{aligned}
(p(x)\, y')' + q(x)\, y + \lambda\, r(x)\, y &= 0, \qquad a < x < b && (3.324)\\
\alpha_0\, y(a) + \alpha_1\, y'(a) &= 0, && (3.325)\\
\beta_0\, y(b) + \beta_1\, y'(b) &= 0. && (3.326)
\end{aligned}
$$

with p, q, r are bounded on $[a, b]$ (that is the interval $a \leqslant x \leqslant b$), $p, r > 0$ for all $a \leqslant x \leqslant b$, and α_0, α_1 real, not both 0, and β_0, β_1 real, not both 0.

We would like to prove:

- It has nonzero solutions for a countably infinite set of values of λ. These eigenvalues are all real. The set of eigenvalues does not have any limit points.
- For any f having two continuous derivatives on $[a, b]$ and satisfying the boundary conditions, the infinite sum
$$\sum_{n=1}^{\infty} c_n\, \varphi_n \tag{3.327}$$
where
$$c_n = \frac{\int_a^b f(x)\, \varphi_n(x)\, r(x)\, \mathrm{d}x}{\int_a^b \varphi_n(x)^2\, r(x)\, \mathrm{d}x} \tag{3.328}$$
converges absolutely uniformly to $f(x)$. By "absolutely uniformly" we mean
$$\sum_1^{\infty} |c_n|\, |\varphi_n| < \infty \tag{3.329}$$
and the convergence to f is uniform.
- The only continuous function f on $[a, b]$ with $\int_a^b f(x)\, \varphi_n(x)\, r(x)\, \mathrm{d}x = 0$ for all n is $f \equiv 0$.
- If φ_n's are chosen such that
$$\int_a^b \varphi_n(x)^2\, r(x)\, \mathrm{d}x = 1 \tag{3.330}$$
We have the following Parseval-type relation
$$\int_a^b f(x)^2\, r(x)\, \mathrm{d}x = \sum_{n=1}^{\infty} |c_n|^2. \tag{3.331}$$

To do these we need to first transform the equation to an equivalent integral equation.

3.3.1. Equivalent integral equation.

First we show that there is a function $G(x; \xi)$ satisfying: $y = \int_a^b G(x, \xi)\, f(\xi)\, \mathrm{d}\xi$ solves

$$
\begin{aligned}
(p(x)\, y')' + q(x)\, y + f(x) &= 0, \qquad a < x < b && (3.332)\\
\alpha_0\, y(a) + \alpha_1\, y'(a) &= 0, && (3.333)\\
\beta_0\, y(b) + \beta_1\, y'(b) &= 0. && (3.334)
\end{aligned}
$$

Such a function is called the "Green's function" to the problem. We do this through explicit construction.

The basic idea is as follows. We try to find a solution to

$$
\begin{aligned}
(p(x)\, y')' + q(x)\, y + \delta(x - \xi) &= 0, \qquad a < x < b && (3.335)\\
\alpha_0\, y(a) + \alpha_1\, y'(a) &= 0, && (3.336)\\
\beta_0\, y(b) + \beta_1\, y'(b) &= 0. && (3.337)
\end{aligned}
$$

and call this solution $G(x, \xi)$. Then it is clear that $y = \int_a^b G(x, \xi)\, f(\xi)\, \mathrm{d}\xi$ is the solution to the original problem. The proof of this is left as an exercise.

Since $\delta(x-\xi)=0$ for $x\neq\xi$, we see that (Here we fix ξ, so $'$ stands for x derivative)

$$(p(x)\,G')'+q(x)\,G=0 \qquad a<x<\xi, \qquad \xi<x<b. \tag{3.338}$$

Now by the theory of 2nd order ODE, there is $y_1(x)$ satisfying

$$(p(x)\,y')'+q(x)\,y=0, \qquad \alpha_0\,y(a)+\alpha_1\,y'(a)=0 \tag{3.339}$$

and $y_2(x)$ satisfying

$$(p(x)\,y')'+q(x)\,y=0, \qquad \beta_0\,y(b)+\beta_1\,y'(b)=0 \tag{3.340}$$

We assume y_1, y_2 are linearly independent. Then we know the general solution to $(p\;y')'+q\;y=0$ is $C_1\,y_1+C_2\,y_2$. Therefore

$$G(x,\xi)=\begin{cases} C_1(\xi)\,y_1(x)+C_2(\xi)\,y_2(x) & a<x<\xi \\ D_1(\xi)\,y_1(x)+D_2(\xi)\,y_2(x) & \xi<x<b \end{cases}. \tag{3.341}$$

Since y_1, y_2 are linearly independent, y_2, y_1 cannot satisfy the boundary condition at a, b respectively. Consequently $C_2=D_1=0$.

To accomodate the $\delta(x-\xi)$ term, we integrate, for any $\varepsilon>0$:

$$\int_{\xi-\varepsilon}^{\xi+\varepsilon}\left[(p(x)\,G')'+q(x)\,G\right]\mathrm{d}x+\int_{\xi-\varepsilon}^{\xi+\varepsilon}\delta(x-\xi)\,\mathrm{d}x=0 \tag{3.342}$$

which leads to

$$p(\xi+\varepsilon)\,G'(\xi+\varepsilon,\xi)-p(\xi-\varepsilon)\,G'(\xi-\varepsilon,\xi)+\int_{\xi-\varepsilon}^{\xi+\varepsilon}q(x)\,G\,\mathrm{d}x+1=0. \tag{3.343}$$

Now take the limit $\varepsilon\longrightarrow 0$, As q,G are bounded, the integral term $\longrightarrow 0$. Using (3.341) we reach

$$p(\xi)\,[C_1\,y_1'-D_2\,y_2']=-1. \tag{3.344}$$

Together with the continuity of G at $x=\xi$:

$$C_1\,y_1-D_2\,y_2=0 \tag{3.345}$$

we obtain

$$C_1=\frac{-y_2}{p\,[y_1'\,y_2-y_1\,y_2']}, \qquad D_2=\frac{-y_1}{p\,[y_1'\,y_2-y_1\,y_2']}. \tag{3.346}$$

Thus we obtain

$$G(x,\xi)=\begin{cases} -\dfrac{y_2(\xi)\,y_1(x)}{p\,[y_1'\,y_2-y_1\,y_2']} & a<x<\xi \\ -\dfrac{y_1(\xi)\,y_2(x)}{p\,[y_1'\,y_2-y_1\,y_2']} & \xi<x<b \end{cases}. \tag{3.347}$$

It turns out that $G(x,\xi)=G(\xi,x)$. (See exercise)

Now we turn back to the Sturm-Liouville problem

$$\begin{aligned} (p(x)\,y')'+q(x)\,y+\lambda\,r(x)\,y &= 0, \qquad a<x<b & (3.348)\\ \alpha_0\,y(a)+\alpha_1\,y'(a) &= 0, & (3.349)\\ \beta_0\,y(b)+\beta_1\,y'(b) &= 0. & (3.350) \end{aligned}$$

We see that the solution can be written as

$$y(x)=\lambda\int_a^b G(x,\xi)\,r(\xi)\,y(\xi)\,\mathrm{d}\xi \tag{3.351}$$

Now set $z(x)=r^{1/2}(x)\,y(x)$ and $k(x,\xi)=r(x)^{1/2}\,G(x,\xi)\,r(\xi)^{1/2}$, we reach

$$z(x)=\lambda\int_a^b k(x,\xi)\,z(\xi)\,\mathrm{d}\xi. \tag{3.352}$$

If we denote the operator

$$K[z]:=\int_a^b k(x,\xi)\,z(\xi)\,\mathrm{d}\xi \tag{3.353}$$

then we have

$$K[z] = \mu\, z \tag{3.354}$$

where $\mu = \lambda^{-1}$.

It is clear that: The integral form problem (3.354) and the original Sturm-Liouville problem share eigenfunctions, and have one-to-one correspondence between eigenvalues. Therefore in the following we study the integral form.

Remark 3.17. Note that there may be a problem if $\lambda = 0$ (and $\lambda < 0$ during the following proofs). This is easily fixed. Since all (possible) eigenvalues are bounded below, we can find a number c and change the equation to

$$(p\, y')' + (q - c\, r)\, y + (\lambda + c)\, r\, y = 0. \tag{3.355}$$

Thus the new eigenvalues are $\lambda + c$ which are all positive.

Remark 3.18. Also note that the orthogonality relations

$$\int_a^b y_n(x)\, y_m(x)\, r(x) \mathrm{d}x = 0 \tag{3.356}$$

becomes

$$\int_a^b z_n(x)\, z_m(x)\, \mathrm{d}x = 0 \tag{3.357}$$

since $z_n(x) = r(x)^{1/2}\, y_n(x)$, $z_m(x) = r(x)^{1/2}\, y_m(x)$.

3.3.2. Some notations.

Now we introduce some notations to simplify the following presentation. Later we will see that these can be greatly generalized.

- Inner product: Let $z, \tilde{z}$ be two real functions, then their "inner product" is defined as

$$(z, \tilde{z}) := \int_a^b z(x)\, \tilde{z}(x)\, \mathrm{d}x. \tag{3.358}$$

  Inner product is the infinite dimensional generalization of dot product for vectors.

- L^2 Norm: We define the L^2 norm as

$$\|z\| = (z, z)^{1/2}. \tag{3.359}$$

  Remark 3.19. Norms are generalizations of length of vectors, which in turn generalizes absolute value for numbers. Thus a basic requirement should be $\|z\| \geqslant 0$, and $\|z\| = 0 \Longleftrightarrow z = 0$. This explains the requirement $r(x) > 0$.

- Orthogonality: Two functions $z, \tilde{z}$ are said to be "orthogonal" if $(z, \tilde{z}) = 0$.

- Linear operators and self-adjointness.

  A linear operator is a mapping $K: L^2 \mapsto L^2$ such that

$$K(a\, z_1 + b\, z_2) = a\, K[z_1] + b\, K[z_2]; \tag{3.360}$$

  The operator is self-adjoint if

$$(K z_1, z_2) = (z_1, K z_2). \tag{3.361}$$

  Self-adjoint operator is a generalization of symmetric (Hermitian if complex) matrices.

- Norm of operators.

  The norm of an operator is defined as

$$\|K\| := \sup_{\|f\| = 1} \|K f\| = \sup_{f \neq 0} \frac{\|K f\|}{\|f\|}. \tag{3.362}$$

An operator is said to be bounded if its norm is finite. From definition we have, for any f,

$$\|K f\| \leqslant \|K\| \, \|f\|. \tag{3.363}$$

- Cauchy-Schwarz inequality. Let $(\cdot, \cdot)$ be an inner product, and $\|\cdot\|$ the associated norm. Then

$$|(f, g)| \leqslant \|f\| \, \|g\|. \tag{3.364}$$

We can show that our operator

$$K f := \int_a^b k(x, \xi) \, f(\xi) \, \mathrm{d}\xi \tag{3.365}$$

is a linear, self-adjoint, bounded operator.

3.3.3. Eigenvalues are countable and isolated.

In this section we show that the eigenvalues are countable and have no limit point(s), in other words, the eigenvalues (assuming it's bounded below) can be listed $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$.

From the duscussion in the last section we see that it suffices to show that the eigenvalues of (3.354) can be listed as $\mu_1 > \mu_2 > \cdots > 0$, with 0 the only possible limit point.

This is done in several steps.

1. There is at least one eigenvalue.

   Consider $m := \sup_{\|z\| = 1} (K[z], z)$ where the supreme is taken over all continuous functions with $\int_a^b z^2 = 1$. Since $k(x, \xi)$ is a bounded function, $m \in \mathbb{R}$. We show that $m = \mu_1$.

   By definition of sup there is $\{z_n\}$ satisfying $\|z_n\| = 1$ and $(K[z_n], z_n) \longrightarrow m$. All we need to do is to show that there is a subsequence of z_n converging uniformly. However this turns out to be not easy. So we apply the following trick: Instead of showing $z_n \rightarrow$ an eigenfunction, we show $K[z_n] \rightarrow$ an eigenfunction.

   - The idea is to try to apply the following

     Theorem 3.20. (Arzela-Ascoli) *Let $\{f_n(x)\}$ be a sequence of functions that are*

     - *Uniformly bounded: There is $M > 0$ such that $|f_n(x)| \leqslant M$ for all $n \in \mathbb{N}$ and all $x \in [a, b]$.*
     - *Equicontinuous: For any $\varepsilon > 0$, there is $\delta > 0$ such that $|x_1 - x_2| < \delta \Longrightarrow |f_n(x_1) - f_n(x_2)| < \varepsilon$ holds for all $n \in \mathbb{N}$ and all $x_1, x_2 \in [a, b]$.*

     Then there is a subsequence $f_{n_k}(x)$ converging uniformly to some continuous function $f(x)$. That is for any $\varepsilon > 0$, there is $K \in \mathbb{N}$ such that for all $k > N$, and all $x \in [a, b]$, $|f_{n_k}(x) - f(x)| < \varepsilon$.

   - First we show that $K[z_n]$ is uniformly bounded. This follows from the inequality

     $$|K[z_n]| \leqslant \int_a^b |k(x, \xi)| \, |z_n(\xi)| \, \mathrm{d}\xi \leqslant \left(\int_a^b k(x, \xi)^2 \, \mathrm{d}\xi \right)^{1/2} \left(\int_a^b |z_n(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2} \tag{3.366}$$

     and the boundedness of k.

   - Next we show that $K[z_n]$ is equicontinuous, that is for any $\varepsilon > 0$ there is $\delta > 0$ such that whenever $|x_1 - x_2| < \delta$, $|K[z_n](x_1) - K[z_n](x_2)| < \varepsilon$.

     To see this we write

     $$\begin{aligned} |K[z_n](x_1) - K[z_n](x_2)| &= \left| \int_a^b [k(x_1, \xi) - k(x_2, \xi)] \, z_n(\xi) \, \mathrm{d}\xi \right| \\ &\leqslant \int_a^b |k(x_1, \xi) - k(x_2, \xi)| \, |z_n(\xi)| \, \mathrm{d}\xi \\ &\leqslant \left(\int_a^b |k(x_1, \xi) - k(x_2, \xi)|^2 \, \mathrm{d}\xi \right)^{1/2} \left(\int_a^b |z_n(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2} \\ &= \left(\int_a^b |k(x_1, \xi) - k(x_2, \xi)|^2 \, \mathrm{d}\xi \right)^{1/2}. \end{aligned} \tag{3.367}$$

Since k is continuous over $[a, b] \times [a, b]$, it is uniformly continuous, thus there is $\delta > 0$ such that whenever $|x_1 - x_2| < \delta$, $|k(x_1, \xi) - k(x_2, \xi)| < \frac{\varepsilon}{\sqrt{b-a}}$.

- Therefore there is a subsequence z_{n_k} such that $K[z_{n_k}]$ converges uniformly to some function φ. We now show that φ must be an eigenfunction with eigenvalue m. We have

$$\begin{aligned} \|K[z_{n_k}] - m\, z_{n_k}\|^2 &= (K[z_{n_k}] - m\, z_{n_k}, K[z_{n_k}] - m\, z_{n_k}) \\ &= (K[z_{n_k}], K[z_{n_k}]) - 2\, m\, (K[z_{n_k}], z_{n_k}) + m^2 \\ &\longrightarrow \|\varphi\|^2 - m^2 \end{aligned} \tag{3.368}$$

This implies $\|\varphi\|^2 \geqslant m^2 > 0 \Longrightarrow \varphi \neq 0$.

Since K is bounded, to show $K\varphi = m\, \varphi$, all we need

$$0 = \|K\varphi - m\, \varphi\| = \lim_{k \longrightarrow \infty} \|K[K[z_{n_k}]] - m\, K[z_{n_k}]\| \Longleftarrow \|K[z_{n_k}] - m\, z_{n_k}\| \longrightarrow 0. \tag{3.369}$$

We re-visit

$$\|K[z_{n_k}] - m\, z_{n_k}\|^2 = (K[z_{n_k}], K[z_{n_k}]) - 2\, m\, (K[z_{n_k}], z_{n_k}) + m^2 \leqslant m^2 \tag{3.370}$$

All we need to show is $\|K[z_{n_k}]\| \leqslant m$.

- Proof of $\|K f\| \leqslant m\, \|f\|$. It suffices to prove $\|K f\| \leqslant m$ for all $\|f\| \leqslant 1$.

Take any f, g with $\|f\| = \|g\| = 1$. We have $(K(f - g), (f - g)) \geqslant 0$ (see exercise) which leads to

$$(K f, f) + (K g, g) - 2\, (K f, g) \geqslant 0 \Longrightarrow (K f, g) \leqslant m. \tag{3.371}$$

This holds for all f, g with $\|f\| = \|g\| = 1$. Now take $g = \frac{K f}{\|K f\|}$ leads to $\|K f\| \leqslant m$. Thus $\|K[z_{n_k}]\| \leqslant m$ as desired.

2. The eigenvalues cannot have nonzero limit point(s).

Assume the contrary. We have

$$K[z_{n_j}] = \lambda_j\, z_{n_j} \tag{3.372}$$

with $\lambda_j \longrightarrow \lambda > 0$. Then without loss of generality, we can assume $\lambda_j \geqslant \lambda/2$ for all j. Now since z_{n_j} are orthogonal to one another, we have

$$\|K[z_{n_j}] - K[z_{n_k}]\| = \lambda_j^2 + \lambda_k^2 \geqslant \lambda^2/2. \tag{3.373}$$

On the other hand we know that there is a subsequence converging uniformly. Contradiction.

3.3.4. Generalized Fourier expansion.

In this section we show that the eigenfunctions can be used to expand a general twice continuously differentiable function.

- First we show that if there is a continuous function f such that $(f, z_n) = \int_a^b f(x)\, z_n(x)\, \mathrm{d}x = 0$ for all eigenfunctions z_n, then $f(x) = 0$.

Assume not. Then consider all such functions. Name this set by H. Without loss of generality, we assume z_n's are normalized, that is $\|z_n\| = 1$.

We first show that if $f \in H$ then so does $K f$:

$$(K f, z_n) = (f, K z_n) = \lambda_n\, (f, z_n) = 0. \tag{3.374}$$

Note that we have used the self-adjointness of K.

Next we show that if $f_m \in H$, $\|f_m - f\| \longrightarrow 0$, then $f \in H$. To see this we use Cauchy-Schwarz:

$$|(f, z_n)| = |(f_m - f, z_n)| \leqslant \|f_m - f\|\, \|z_n\| = \|f_m - f\| \longrightarrow 0. \tag{3.375}$$

Therefore we can repeat the argument of existence of eigenvalues to show that $\sup_{f \in H, \|f\| = 1} (K f, f) > 0$ (because $(K f, f) > 0$ for all $f \neq 0$) is an eigenvalue with some eigenfunction $\tilde{f}$. But then there must be n such that $\tilde{f} = z_n$ which leads to $\left(\tilde{f}, z_n\right) \neq 0$, contradiction.

- Next we show the Parseval relation. By Bessel's inequality, we have the convergence of $\sum c_n z_n$ with $c_n = \frac{(f, z_n)}{\|z_n\|^2}$. Now $f - \sum c_n z_n$ satisfies

$$\left(f - \sum c_n z_n, z_n\right) = 0 \tag{3.376}$$

thus we have $f = \sum c_n z_n$ which leads to Parseval's equality.

- Finally we show that if $f \in C^2$ (and satisfies the boundary conditions) then the expansion converges uniformly. Notice that every such f can be written as $K[g]$ with g continuous. Then

$$\sum_{n=M}^{N} |(f, z_n) z_n| = \sum_{n=M}^{N} \mu_n |g_n z_n| \longrightarrow 0 \tag{3.377}$$

uniformly because $\mu_n \longrightarrow 0$. This means the sequence $\sum_{n=M}^{N} (f, z_n) z_n$ is Cauchy (uniformly in x) and convergence follows.

Exercises.

Exercise 3.14. Show that if $G(x, \xi)$ solves

$$\begin{aligned} (p(x) y')' + q(x) y + \delta(x - \xi) &= 0, \qquad a < x < b & (3.378)\\ \alpha_0 y(a) + \alpha_1 y'(a) &= 0, & (3.379)\\ \beta_0 y(b) + \beta_1 y'(b) &= 0. & (3.380) \end{aligned}$$

then $y = \int_a^b G(x, \xi) f(\xi) \mathrm{d}\xi$ solves

$$\begin{aligned} (p(x) y')' + q(x) y + f(x) &= 0, \qquad a < x < b & (3.381)\\ \alpha_0 y(a) + \alpha_1 y'(a) &= 0, & (3.382)\\ \beta_0 y(b) + \beta_1 y'(b) &= 0. & (3.383) \end{aligned}$$

Exercise 3.15. Calculate Green's functions for the following problems.

a) $y'' = 0, y(0) = y(1) = 0.$

b) $y'' = 0,\ y(0) = y'(1) = 0.$

Exercise 3.16. In the construction of Green's function, we need the solutions to

$$(p(x) y')' + q(x) y = 0, \qquad \alpha_0 y(a) + \alpha_1 y'(a) = 0 \tag{3.384}$$

and

$$(p(x) y')' + q(x) y = 0, \qquad \beta_0 y(b) + \beta_1 y'(b) = 0 \tag{3.385}$$

to be linearly independent. What happens if that's not the case?

Exercise 3.17. Prove that the Green's function $G(x, \xi)$ as defined in (3.347) is symmetric: $G(x, \xi) = G(\xi, x)$. (Hint: Show that $p[y_1' y_2 - y_1 y_2']$ is constant).

Exercise 3.18. Show K is linear, self-adjoint, bounded.

Exercise 3.19. Let K be defined as

$$K f := \int_a^b k(x; \xi) f(\xi) \mathrm{d}\xi \tag{3.386}$$

where

$$k(x; \xi) = r(x)^{1/2} G(x; \xi) r(\xi)^{1/2} \tag{3.387}$$

with $G(x; \xi)$ the Green's function for the operator

$$-(p y')' + q y \text{ with boundary conditions } y(a) = y(b) = 0 \tag{3.388}$$

in the sense that the solution to

$$-(p y')' + q y = f, \qquad y(a) = y(b) = 0 \tag{3.389}$$

is given by

$$y(x) = \int_a^b G(x; \xi) f(\xi) \mathrm{d}\xi. \tag{3.390}$$

Assume $p, q \geqslant 0$. Show that K is a non-negative operator, that is $(K z, z) \geqslant 0$ for all continuous functions z.

References.

- Anthony W. Knapp, "Advanced Real Analysis", Chap. 2.
- Kôsaku Yosida, "Lectures on Differential and Integral Equations", Chap. 2.

3.4. Higher Dimensional Problems.

In contrast to Section 3.1, here we consider higher dimensional equations in an irregular domain instead of regular ones like rectangle, disc, sphere, etc. Note that the shared property of those "regular" domains is that they are either rectangular or can be tranformed to a rectangular domain through change of variables. In this case one can still formulate the eigenvalue problems and study their properties, but usually one cannot write explicit formulas for the eigenfunctions anymore.

3.4.1. Higher dimensional separation of variables.

In the following we use the notation $\boldsymbol{x} = (x_1, \ldots, x_n)$. We consider the following equation in higher dimensions: Let Ω be a domain in $\mathbb{R}^n$, let L be a linear differential operator of the following form:

$$L[u] = -\sum_{i,j=1}^{n} (p(\boldsymbol{x})\, u_{x_j})_{x_i} + q(\boldsymbol{x})\, u = -\nabla \cdot (p\, \nabla u) + q\, u \tag{3.391}$$

For example, when $p = 1$, $q = 0$, L is the usual Laplace operator

$$L[u] = -\triangle u. \tag{3.392}$$

We will the reason for the negative sign later.

Then we set up the equation as

$$u_t + L[u] = 0 \qquad \Omega \times \mathbb{R}^+, \qquad u(\boldsymbol{x}, 0) = f(\boldsymbol{x}) \tag{3.393}$$

with certain boundary conditions.

Now we can apply the idea of separation of variables. Since Ω is a general domain, there is usually no change of variables to map it to a rectangular one, therefore we can only write $T(t)\, X(\boldsymbol{x})$ and try to require it to solve the equation. Substituting into the equation we reach

$$T'(t)\, X(x) + T(t)\, L[X] = 0 \Longrightarrow L[X] - \lambda\, X = 0, \qquad T'(t) + \lambda\, T(t) = 0. \tag{3.394}$$

From our understanding of the method, we expect the eigenvalue problem

$$L[X] - \lambda\, X = 0, \qquad \text{boundary conditions} \tag{3.395}$$

to have countably many eigenvalues which can be listed by their sizes: $\lambda_1 < \lambda_2 < \cdots$ and for each eigenvalue we expect to have one $X_i(\boldsymbol{x})$. These $X_i(\boldsymbol{x})$ are orthogonal in the sense that

$$\int_\Omega X_n(\boldsymbol{x})\, X_m(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0 \qquad m \neq n \tag{3.396}$$

and complete in the sense that for any reasonably smooth $f(\boldsymbol{x})$, we can write

$$f(\boldsymbol{x}) = \sum_{n=1}^{\infty} f_n\, X_n(\boldsymbol{x}) \text{ with } f_n = \frac{\int_\Omega f(\boldsymbol{x})\, X_n(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}}{\int_\Omega X_n(\boldsymbol{x})^2\, \mathrm{d}\boldsymbol{x}}. \tag{3.397}$$

Then the final solution should be

$$u(\boldsymbol{x}, t) = \sum_{n=1}^{\infty} f_n\, e^{-\lambda_n t}\, X_n(\boldsymbol{x}). \tag{3.398}$$

Before we proceed, we have to point out that some of the above expectations are not realistic. For example, in higher dimensions, it may happen that more than one linearly independent eigenfunctions correspond to one same eigenvalue λ. To see this, recall the heat equation in rectangular domain

$$u_t = u_{xx} + u_{yy}, \qquad 0 < x < \pi, 0 < y < \pi. \tag{3.399}$$

We know that the eigenvalues are $m^2 + n^2$ with eigenfunctions $\sin n\, x \sin m\, x$. But clearly there is the possibility that $m_1^2 + n_1^2 = m_2^2 + n_2^2$.

Now we introduce the boundary conditions:

$$\alpha(\boldsymbol{x})\, u + \beta(\boldsymbol{x})\, \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega. \tag{3.400}$$

This looks complicated but is in fact a natural generalization of the 1D boundary conditions

$$\alpha_1\, u(a,t) + \beta_1\, u_x(a,t) = \alpha_2\, u(b,t) + \beta_2\, u_x(b,t) = 0. \tag{3.401}$$

Example 3.21. Consider the heat equation in cylinder (Example 3.12). The eigenfunctions are $X_{n,k,1}(x, y) = R_{n,k}(r)\cos(n\,\theta)$, $X_{n,k,w}(x,y) = R_{n,k}\sin(n\,\theta)$. Where $R_{n,k}(r) = J_n(\alpha_{n,k}\, r)$ with J_n Bessel function of the first kind. It is important to notice that these eigenfunctions are orthogonal:

$$\int_{x^2+y^2\leqslant 1} X_{n,k,i}(x,y)\, X_{m,l,j}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{3.402}$$

unless $n = m, k = l, i = j$. Note that the weight r for the orthogonality relation of J_n follows from change of variables to polar coordinates.

As in the 1D case, we turn to a more abstract setting. We define the following inner product:

$$(u,v) := \int_\Omega u(\boldsymbol{x})\, v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \tag{3.403}$$

which induces to the norm

$$\|u\| := \left(\int_\Omega u(\boldsymbol{x})^2\,\mathrm{d}\boldsymbol{x}\right)^{1/2}. \tag{3.404}$$

Recall that in 1D what we actually study is the inverse operator L^{-1} defined through Green's function. Such explicit definition is not possible anymore. Nevertheless, it turns out that L^{-1} is still well-defined under the following conditions:

- There is $p_0 > 0$ such that $p(\boldsymbol{x}) \geqslant p_0$ for all $\boldsymbol{x} \in \Omega$.
- $q(\boldsymbol{x}) \geqslant 0$ for all $\boldsymbol{x} \in \Omega$.
- $\alpha(\boldsymbol{x})\,\beta(\boldsymbol{x}) \geqslant 0$ for all $\boldsymbol{x} \in \partial\Omega$.

The proof of this fact is beyond this course. Let $K = L^{-1}$. We will only explicitly use K in a few places in the following discussions.

Remark 3.22. Under the above assumptions the operator $L = -\nabla\cdot(p\,\nabla) + q$ is "uniformly elliptic". A paradigm uniformly elliptic operator is $-\triangle$. In appropriate setting, uniformly elliptic operators enjoy the same nice properties as $-\triangle$.

Similar to the situation in 1D, a few simple facts can be easily derived.

- The operatos L, K are self-adjoint. We check

$$\begin{aligned}(L[u],v) &= \int_\Omega [-\nabla\cdot(p\,\nabla u) + q\,u]\,v\\ &= \int_\Omega p\,\nabla u\cdot\nabla v + q\,u\,v - \int_{\partial\Omega} p\,n\cdot\nabla u\,v\\ &= \int_\Omega p\,\nabla u\cdot\nabla v + q\,u\,v - \int_{\partial\Omega} p\,v\,\frac{\partial u}{\partial n}\\ &= \int_\Omega [-\nabla\cdot(p\,\nabla v) + q\,v]\,u + \int_{\partial\Omega} p\left[u\,\frac{\partial v}{\partial n} - v\,\frac{\partial u}{\partial n}\right]\\ &= (u, L[v]).\end{aligned} \tag{3.405}$$

  Note that the boundary term vanishes due to the boundary condition (3.400).

  The self-adjointness of K easily follows as

$$(K[u],v) = (K[u], L(K[v])) = (L(K[u]), K[v]) = (u, K[v]). \tag{3.406}$$

  Remark 3.23. We see that after integration by parts twice, $[-\nabla\cdot(p\ \nabla u) + q\ u]\ v$ becomes $[-\nabla\cdot(p\,\nabla v) + q\,v]$. If we ignore the boundary terms, we already have $(L[u],v) = (u, L[v])$. Such L is said to be "formally self-adjoint". More generally, the "formal adjoint" of a differential operator L can be defined through

$$(L[u],v) = (u, L^*[v]) + \text{boundary terms}. \tag{3.407}$$

Thus an operator is self-adjoint if

1. It is formally self-adjoint;

2. The boundary condition is appropriate.

- All eigenvalues (if any) are non-negative.

$$\begin{aligned}\lambda\,\|u\|^2=\lambda\,(u,u) &= (L[u],u)\\ &= \int_\Omega [p\,|\nabla u|^2+q\,u^2]\,\mathrm{d}\boldsymbol{x}-\int_{\partial\Omega} p\,u\,\frac{\partial u}{\partial n}\,\mathrm{d}A\\ &= \int_\Omega [p\,|\nabla u|^2+q\,u^2]\,\mathrm{d}\boldsymbol{x}+\int_{\partial\Omega} p\,\frac{\alpha}{\beta}\,u^2\,\mathrm{d}A\geqslant 0.\end{aligned} \tag{3.408}$$

- Eigenfunctions corresponding to different eigenvalues are orthogonal.

  Let $L[u]=\lambda\,u$, $L[v]=\mu\,v$ with $\lambda\neq\mu$. Then we have

$$\lambda\,(u,v)=(L[u],v)=(u,L[v])=\mu\,(u,v)\Longrightarrow(u,v)=0. \tag{3.409}$$

  Note that as the eigenspaces may not be one-dimensional, we cannot yet conclude the existence of an orthogonal system of eigenfunctions ordered by their corresponding eigenvalues. To do that we need two things:

  1. For each eigenspace we can find a orthogonal basis;

  2. Each eigenspace is finite dimensional.

  The first can be done using the following Gram-Schmidt orthogonalization procedure, the second needs some deeper properties of compact operators.

3.4.2. Gram-Schmidt orthogonalization.

Recall that now it is possible to have more than one linearly independent eigenfunctions corresponding to one single eigenvalue. Then there is no way to prove that they must be orthogonal. However, given a set of linearly independent functions, we can always "orthogonalize" them through the following Gram-Schmidt process.

Let $u_1,\ldots,u_n$ be a set of linearly independent functions. We define

$$v_1 = \frac{u_1}{\|u_1\|}; \tag{3.410}$$

$$v_2 = \frac{u_2-(u_2,v_1)\,v_1}{\|u_2-(u_2,v_1)\,v_1\|}; \tag{3.411}$$

$$v_3 = \frac{u_3-(u_3,v_1)\,v_1-(u_3,v_2)\,v_2}{\|u_3-(u_3,v_1)\,v_1-(u_3,v_2)\,v_2\|}; \tag{3.412}$$

$$\vdots \qquad \vdots$$

It is easy to see that $v_1,\ldots,v_n$ now form a set of orthonormal functions, that is

$$\|v_i\|=1,\qquad (v_i,v_j)=0 \text{ when } i\neq j. \tag{3.413}$$

We also note that

$$\operatorname{span}\{v_1,\ldots,v_k\}=\operatorname{span}\{u_1,\ldots,u_k\} \tag{3.414}$$

for all $1\leqslant k\leqslant n$.

3.4.3. Hilbert spaces, compact operators.

A Hilbert space is a special kind of abstract vector space.

A vector space is a set V with two operations defined: addition between elements of V and multiplication between an element of V and a number α. These two operations share the properties of the same operations on $\mathbb{R}^n$ and $\mathbb{R}$:

i. There is an element $\mathbf{0}\in V$ such that $\mathbf{0}+\boldsymbol{v}=\boldsymbol{v}$ for all $\boldsymbol{v}\in V$;

ii. $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$;

iii. $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$;

iv. For each $\boldsymbol{u}\in V$ there is an element $-\boldsymbol{u}\in V$ such that $\boldsymbol{u}+(-\boldsymbol{u})=\boldsymbol{0}$.

v. $1\cdot\boldsymbol{u}=\boldsymbol{u}$ for all $\boldsymbol{u}\in V$;

vi. $(a+b)\,\boldsymbol{u}=a\cdot\boldsymbol{u}+b\cdot\boldsymbol{u}$;

vii. $a\,(\boldsymbol{u}+\boldsymbol{v})=a\cdot\boldsymbol{u}+b\cdot\boldsymbol{u}$;

viii. $a\,(b\,\boldsymbol{u})=(a\,b)\,\boldsymbol{u}$.

Remark 3.24. The above is called "real" vector space since the scalars a, $b\in\mathbb{R}$. We can replace $\mathbb{R}$ by the complex numbers $\mathbb{C}$ or any other fields to obtain other types of vector spaces. Some of the following definitions, in particular the definition of inner product, may need to be modified in those cases.

Example 3.25. The set of continuous functions with usual addition between functions and scalar-function multiplication is a vector space.

Now we define a linear operator. A linear operator is a mapping L between two vector spaces V, W satisfying

i. $L[\boldsymbol{v}+\boldsymbol{w}]=L[\boldsymbol{v}]+L[\boldsymbol{w}]$;

ii. $L[a\,\boldsymbol{v}]=a\,L[\boldsymbol{w}]$.

Example 3.26. Let V be the set of continuously differentiable functions and W the set of continuous functions, then $L=\frac{\mathrm{d}}{\mathrm{d}x}$ is a linear operator from V to W.

Note that all we can do in an abstract vector space is linear algebra. Analysis is not possible because there is no definition of convergence yet. To be able to discuss convergence, we need to introduce the idea of "norm". This leads to the definition of Banach space, to which Hilbert space is a special case. We jump directly to this special case.

Definition 3.27. *A Hilbert space H is an abstract vector space with an "inner product" defined. An inner product is a bilinear form $(\cdot,\cdot)\colon H\times H\mapsto\mathbb{R}$ satisfying*

i. $(u,v)=(v,u)$*;*

ii. $(a\,u+b\,v,w)=a\,(u,w)+b\,(v,w)$*;* $(u,a\,v+b\,w)=a\,(u,v)+b\,(u,w)$*.*

iii. $(u,u)\geqslant 0$*, with equality if and only if* $u=0$*.*

Example 3.28. The set of all square integrable functions with inner product $(u,v)=\int\,u\,v\,\mathrm{d}x$ is a Hilbert space.

With inner product comes the norm:

$$\|u\|:=(u,u)^{1/2}. \tag{3.415}$$

With norm comes convergence:

$$u_n\longrightarrow u \text{ if } \|u_n-u\|\longrightarrow 0. \tag{3.416}$$

and boundedness of linear operators: L is bounded if there is $K>0$ such that

$$\|L\,u\|\leqslant K\,\|u\|,\qquad \forall u\in H. \tag{3.417}$$

With convergence comes the continuity of linear operators: A linear operator L on a Hilbert space H is continuous if $u_n\longrightarrow u$ then $L u_n\longrightarrow L u$.

Definition 3.29. **(Compact operator)** *An operator on a Hilbert space $L\colon H\mapsto H$ is compact if $\{u_n\}$ bounded $\Longrightarrow$ there is a subsequence u_{n_k} such that $L u_{n_k}$ converges.*

Example 3.30. If H is finite-dimensional then all linear operators on it are compact; When H is infinite-dimensional, there are non-compact linear operators. A paradigm example is the identity operator $L u = u$. Another is the shift operator:

$$(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots) \tag{3.418}$$

on the Hilbert space of infinite dimensional vectors with inner product:

$$(\boldsymbol{x}, \boldsymbol{y}) = \sum_{n=1}^{\infty} x_n y_n. \tag{3.419}$$

3.4.4. Higher dimensional Sturm-Liouville type theory.

In this section we try to prove the following properties of the eigenvalue problem

$$-\nabla \cdot (p \nabla u) + q u = \lambda u, \qquad \alpha(\boldsymbol{x}) u + \beta(\boldsymbol{x}) \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega. \tag{3.420}$$

a) There is at least one eigenvalue.

Define the energy functional $E[u] := (L[u], u) = \int p |\nabla u|^2 + q u^2 + \int_{\partial\Omega} p \frac{\alpha}{\beta} u^2$.

We claim that,

$$\lambda_1 = \min_{u \neq 0} \frac{(L[u], u)}{(u, u)}. \tag{3.421}$$

The fact that this minimum is attained requires knowledge of Sobolev spaces and is omitted. Now we assume u_1 is the minimizer, and try to show that it is an eigenfunction. Consider a small parametrized perturbation

$$\frac{(L[u + s v], u + s v)}{(u + s v, u + s v)} = \frac{(L[u], u) + 2 s (L[u], v) + s^2 (L[v], v)}{(u, u) + 2 s (u, v) + s^2 (v, v)}. \tag{3.422}$$

Since $s = 0$ is a minimizer, taking derivative $\frac{\mathrm{d}}{\mathrm{d}s}$ we reach

$$[2 (L[u], v) + 2 s (L[v], v)] [(u, u) + 2 s (u, v) + s^2 (v, v)] - (L[u], u) + 2 s (L[u], v) + s^2 (L[v], v) [2 (u, v) + 2 s (v, v)] = 0. \tag{3.423}$$

Now set $s = 0$ we reach

$$(L[u], v) (u, u) = (L[u], u) (u, v) \Longrightarrow (L[u_1] - \lambda_1 u_1, v) = 0. \tag{3.424}$$

Since this is true for all v, we conclude that $L[u_1] = \lambda_1 u_1$.

That λ_1 is the smallest eigenvalue is trivial.

b) There are countably many eigenvalues which can be listed $\lambda_1 < \lambda_2 < \ldots$, and each eigenspace is finite dimensional.

Note that we can obtain $\lambda_2, \lambda_3, \ldots$ as follows:

$$\lambda_2 = \min_{u \perp u_1} \frac{(L[u], u)}{(u, u)}, \qquad \lambda_3 = \min_{u \perp u_1, u_2} \frac{(L[u], u)}{(u, u)} \tag{3.425}$$

Note that here it may happen that $\lambda_2 = \lambda_1$.

This way we obtain countably many eigenvalues.

To show that each eigenvalue is only repeated finitely many times we need to turn to the inverse operator $K = L^{-1}$ and try to show that the eigenspace of each of its eigenvalues is finite dimensional. An important fact we have to use is that K is a compact operator. Thus we show: If K is compact and λ is an eigenvalue, then there are only finitely many linearly independent eigenvectors. Assume the contrary, then there is a sequence of linearly independent eigenvectors $u_1, u_2, \ldots$. Apply the Gram-Schmidt orthogonalization procedure, we obtain $v_1, v_2, \ldots$ such that

$$(v_i, v_j) = \delta_{ij}. \tag{3.426}$$

Now it is easy to show that $\{K v_i\}$ does not have any convergent subsequence. Contradiction.

To show that these are all the eigenvalues we only need to show that if any function f is perpendicular to all these eigenfunctions, then $f = 0$, that is the eigenfunctions form a complete set of the space $L^2(\Omega)$.

c) The normalized eigenfunctions form a complete orthonormal set.

To do this we need to show first that $\lambda_n \longrightarrow \infty$. Assume the contrary, that is $\lambda_n \longrightarrow \lambda \in \mathbb{R}$ (note that λ_n is increasing, so this is exactly the contrary to $\lambda_n \longrightarrow \infty$). Take one eigenfunction u_n for each λ_n and normalize it, we obtain a sequence $\{u_n\}$ such that

$$(u_i, u_j) = \delta_{ij}, \qquad K u_i = \lambda_i u_i. \tag{3.427}$$

Since $\lambda_i \longrightarrow \lambda$, there is $N \in \mathbb{N}$ such that for all $n > N$, $\lambda/2 < \lambda_n \leqslant \lambda$. Then we have

$$\|K u_i - K u_j\|^2 = \|K u_i\|^2 + \|K u_j\|^2 > \frac{\lambda^2}{2} \tag{3.428}$$

which contradicts the compactness of K.

Now note that if $u \perp u_1, \ldots, u_N$, then we have

$$(L[u], u) \geqslant \lambda_N (u, u) \Longrightarrow \|u\| \leqslant \frac{1}{\lambda_N} (L[u], u). \tag{3.429}$$

But this leads to contradiction unless $(L[u], u) = \infty$!

Remark 3.31. We notice that the convergence proof seems simpler than in the 1D case. The reason is that here we only proved the convergence in L^2 norm instead of uniform convergence.

Exercises.

Exercise 3.20. Prove that for a linear operator, the following are equivalent:

a) It is continuous;

b) It is continuous at 0;

c) It is bounded.

Exercise 3.21. Let H be a finite dimensional Hilbert space, and L is a linear operator $L: H \mapsto H$. Prove that

a) L is bounded.

b) L is compact.

Exercise 3.22. Construct a sequence $f_n(x) \longrightarrow 0$ for every $x \in \mathbb{R}$, but $\|f_n\| = \left(\int_{\mathbb{R}} f(x)^2 \, \mathrm{d}x\right)^{1/2} = 1$ for all n.

Exercise 3.23. Let V be a linear vector space. A norm $\|\cdot\|$ is a mapping $V \mapsto \mathbb{R}$ satisfying

a) For any $v \in V$, $\|v\| \geqslant 0$, and $\|v\| = 0 \Longleftrightarrow v = 0$.

b) For any $v \in V$ and $a \in \mathbb{R}$, $\|a v\| = |a| \, \|v\|$.

c) For any $u, v \in V$, $\|u + v\| \leqslant \|u\| + \|v\|$.

Prove that

$$\|v\| := \sup_{x \in [a,b]} |f(x)| \tag{3.430}$$

is a norm on $V = \{f(x): [a,b] \mapsto \mathbb{R} | \, f(x) \text{ is bounded}\}$. Then show that this norm does not come from an inner product, that is there can be no inner product that $\|v\|^2 = (v, v)$. (Hint: Show that if $(\cdot, \cdot)$ is an inner product, then $(u + v, u + v) + (u - v, u - v) = 2 (u, u) + 2 (v, v)$.)

Exercise 3.24. Recall the Legendre's polynomials are eigenfunctions of

$$(1 - x^2) \, y'' - 2 x y' + \lambda y = 0, \qquad -1 < x < 1 \tag{3.431}$$

satisfying: $P_m(x)$ is a polynomial of degree m.

Prove: If we take $u_m(x) = x^m$ and apply Gram-Schmidt orthogonalization to them using inner product $(u, v) = \int_{-1}^{1} u(x) \, v(x) \, \mathrm{d}x$, the resulting orthonormal set is $\{a_m P_m(x)\}$, where $a_m = \left(\int_{-1}^{1} [P_m(x)]^2 \, \mathrm{d}x\right)^{-1/2}$.

3.5. Problems.

3.5.1. Frobenius theory of power series solutions.

One way to understand a general linear second order equation

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{3.432}$$

is through the power series method, which in its simplest form works as follows:

1. Write $y = \sum_{n=0}^{\infty} a_n\, (x - x_0)^n$,
2. Substitute this y into the equation, obtain formulas for a_n.
3. Study the resulting infinite series $\sum_{n=0}^{\infty} a_n\, (x - x_0)^n$ (now all a_n's are known).

However, this simple approach only works when both $p(x)$ and $q(x)$ are analytic at x_0 (such x_0 is called "ordinary"). Recall that a function $f(x)$ is analytic at a point x_0 if there is $\rho > 0$ such that

$$f(x) = \text{Its Taylor expansion at } x_0 \qquad \forall |x - x_0| < \rho. \tag{3.433}$$

All infinitely differentiable (at x_0) functions have a Taylor expansion, but not all infinitely differentiable functions are analytic, as can be seen from the example $f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leqslant 0 \end{cases}$ whose Taylor expansion at $x_0 = 0$ is

$$0 + 0 \cdot x + \frac{0}{2} \cdot x^2 + \frac{0}{6} \cdot x^3 + \cdots = 0 \neq f(x) \tag{3.434}$$

for every $x > 0$.

When $p(x), q(x)$ are not both analytic at x_0 (in the following for simplicity of presentation we set $x_0 = 0$ when writing expansions), the point x_0 is called "singular". the power series method can still be adapted to work when x_0 is "regular singular", that is (remember we set $x_0 = 0$ when writing formulas)

$$p(x)\, x, \qquad q(x)\, x^2 \tag{3.435}$$

are analytic. In other words $p(x)$ has a pole of order at most 1, and $q(x)$ has a pole of order at most 2. The following theorem guaranteed that the power series method still works as long as the starting ansatz is changed to

$$y = x^{\nu} \sum_{n=0}^{\infty} a_n\, (x - x_0)^n \tag{3.436}$$

where ν may be complex.

Theorem 3.32. (Fuchs) *x_0 is regular singular $\Longleftrightarrow$ There is $\rho \in \mathbb{R}$, $\rho > 0$, such that for all solutions $y(x)$ of (3.432),* $\lim_{x \longrightarrow x_0} (x - x_0)^{\rho}\, y(x) = 0$.

Proof.

- $\Longrightarrow$. This part is relatively easy (but tedious). One just needs to start from the ansatz (3.436) and solve the equation, that is figuring out ν as well as all a_n's, and then prove convergence of the resulting series solution. The details can be found in any elementary ODE book such as the textbooks for Math 334 and Math 201 here at UA.
- $\Longleftarrow$. We need to show that if there is $\rho > 0$ such that $\lim_{x \longrightarrow 0} x^{\rho}\, y(x) = 0$ for all solutions $y(x)$, then $p(x)\, x, q(x)\, x^2$ are analytic. This part is a bit tricky. The outline of the proof is as follows.
    1. First show the existence of a solution $y(x) = x^r\, \varphi(x)$ where $\varphi(x)$ is analytic.

       To do this, let y_1, y_2 be linearly independent solutions. Then if we continue them along circles around x_0, the resulting functions Y_1, Y_2, which still solve the equation, can be represented as

       $$Y_1 = \alpha\, y_1 + \beta\, y_2; \qquad Y_2 = \gamma\, y_1 + \delta\, y_2. \tag{3.437}$$

       Thus for any solution $y = c\, y_1 + d\, y_2$, after such continuation we get

       $$Y = \left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \right] \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{3.438}$$

Now let $\begin{pmatrix} c \\ d \end{pmatrix}$ be an eigenvector, then

$$Y = \lambda\, y \tag{3.439}$$

after one round of continuation. Take r_1 by setting

$$e^{2\pi i r_1} = \lambda. \tag{3.440}$$

Note that we need to show $\lambda \neq 0$ through showing $\alpha\,\delta - \beta\gamma \neq 0$.

Thus

$$y = x^{r_1}\,\varphi_0 \tag{3.441}$$

where φ_0 is single valued. As $\lim_{x\longrightarrow 0} x^\rho\, y(x) = 0$ φ_0 can have at most a pole at $x = 0$ which gives

$$y = x^r\,\varphi \tag{3.442}$$

for some $r \in \mathbb{C}$.

2. Now we denote the above solution by y_1. And obtain y_2 through reduction of order. We will get

$$y_2(x) = y_1(x)\,[a \log x + x^s\,\psi(x)] \tag{3.443}$$

where a is a constant and ψ is single valued analytic except at $x = 0$. Similar to φ we see that ψ has at most a pole.

3. Now recall that

$$p(x) = \frac{y_1''\,y_2 - y_2''\,y_1}{y_1\,y_2' - y_2\,y_1'}; \qquad q(x) = -\frac{y_1''}{y_1} - p\,\frac{y_1'}{y_1} \tag{3.444}$$

We obtain the desired result.

□

Reference.

- K. Yosida, “Lectures on Differential and Integral Equations”, §12.