Math 436 Midterm Solution

Oct. 18, 2012 12:30pm – 1:50pm Total 50 Pts.

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Problem 1. (10 pts) Let c, κ be constants, $\kappa > 0$. Design a random walk model which leads to the equation

$$u_t + c\,u_x = \kappa\,u_{xx}, \tag{1}$$

then obtain Duhamel's principle for the corresponding nonhomogeneous equation

$$u_t + c\,u_x = \kappa\,u_{xx} + f(x,t). \tag{2}$$

Solution. Consider a random walk with probability p moving left and $q = 1 - p$ moving right, with spatial step size h and time step size τ. Then we have

$$u(x, t+\tau) = p\,u(x+h,t) + q\,u(x-h,t). \tag{3}$$

Taylor expansion leads to

$$u + u_t\,\tau + o(\tau) = p\left[u + u_x\,h + \frac{1}{2}\,u_{xx}\,h^2 + o(h^2)\right] + q\left[u - u_x\,h + \frac{1}{2}\,u_{xx}\,h^2 + o(h^2)\right] \tag{4}$$

which simplifies to

$$u_t + \frac{q-p}{\tau}\,h\,u_x = \frac{h^2}{2\,\tau}\,u_{xx} + o\left(\frac{h^2}{\tau}\right) + o(1). \tag{5}$$

If we take $\frac{h^2}{2\,\tau} \longrightarrow \kappa$ and $\frac{q-p}{\tau}\,h \longrightarrow c$, we reach the equation

$$u_t + c\,u_x = \kappa\,u_{xx}, \tag{6}$$

For the Duhamel's principle, we consider the situation that at location x, after time t, the probability is increased by $\tau\,f(x,t)$. Then the equation becomes

$$u_t + c\,u_x = \kappa\,u_{xx} + f(x,t). \tag{7}$$

Thus the Duhamel's principle should read:

$$u(x,t) = U(x,t) + \int_0^t v(x,t;s)\,\mathrm{d}s \tag{8}$$

where $U(x,t)$ solves

$$u_t + c\,u_x = \kappa\,u_{xx}, \qquad u(x,0) = u_0(x) \tag{9}$$

and $v(x,t;s)$ solves

$$v_t + c\,v_x = \kappa\,v_{xx}, \qquad v(x,s;s) = f(x,s). \tag{10}$$

Problem 2. (10 pts) Solve the initial value problem

$$x\,u_x + y\,u_y = x\,e^{-u}, \qquad u = 0 \text{ on } y = x^2. \tag{11}$$

Solution. We have

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}u}{x\,e^{-u}}. \tag{12}$$

This leads to

$$y\,\mathrm{d}x - x\,\mathrm{d}y = 0 \Longrightarrow \mathrm{d}(y/x) = 0. \tag{13}$$

and

$$\mathrm{d}x = \frac{\mathrm{d}u}{e^{-u}} = \mathrm{d}(e^u) \Longrightarrow \mathrm{d}(e^u - x) = 0. \tag{14}$$

So the general solution reads

$$e^u = x + f(y/x) \tag{15}$$

where f is an arbitrary function.

Now apply the initial condition:

$$e^0 = x + f(x^2/x) = x + f(x) \Longrightarrow f(x) = 1 - x. \tag{16}$$

Therefore the solution is

$$e^u = x + [1 - y/x] \tag{17}$$

or

$$u = \ln\,[x + 1 - y/x]. \tag{18}$$

Problem 3. (10 pts) Construct entropy solution to the conservation law

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \qquad u(x,0) = g(x) = \begin{cases} 0 & x < -1 \\ x+1 & -1 \leqslant x \leqslant 0 \\ 1-x & 0 < x \leqslant 1 \\ 0 & x > 1 \end{cases}. \tag{19}$$

What is the limiting function $v(x) := \lim_{t \longrightarrow \infty} u(x,t)$?

Solution. Drawing characteristics we see that the first singularity appears at $x = 1, t = 1$. A shock emanates from that point, with speed determined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2}\frac{x+1}{t+1} \Longrightarrow x + 1 = C\sqrt{t+1} \tag{20}$$

Since it passes $x = t = 1$, we have $C = 1$. So the shock is

$$x + 1 = \sqrt{t+1} \tag{21}$$

The solution reads

$$u(x,t) = \begin{cases} 0 & x < -1 \\ \dfrac{x+1}{t+1} & -1 < x < t,\ t \leqslant 1 \\ & \text{ and } -1 < x < \sqrt{t+1} - 1,\ t > 1 \\ \dfrac{x-1}{t-1} & t < x \leqslant 1,\ 0 < t \leqslant 1 \\ 0 & x > 1, t \leqslant 1 \text{ and } x > \sqrt{t+1} - 1, t > 1 \end{cases}. \tag{22}$$

The limit is $v(x) = 0$.

Remark 1. How to get formulas for the solution.

Characteristics starting from $-1 \leqslant x \leqslant 0$ has speed: $\frac{\mathrm{d}x}{\mathrm{d}t} = g(x_0) = x_0 + 1$. Therefore the equations for the characteristics are $x = x_0 + (x_0 + 1)\,t$. Thus the solution in this region is

$$u(x_0 + (x_0 + 1)\,t, t) = u(x,t) = g(x_0). \tag{23}$$

To find $u(x,t)$, we need to represent x_0 using x, t.

$$x = x_0 + (x_0 + 1)\,t \Longrightarrow x_0 = \frac{x - t}{t+1} \Longrightarrow u(x,t) = g(x_0) = \frac{x+1}{t+1}. \tag{24}$$

Problem 4. (10 pts) Find the general solution to the following equation:

$$3\,u_{xx} + 10\,u_{xy} + 3\,u_{yy} = 0. \tag{25}$$

Solution. We have

$$3\,(\mathrm{d}y)^2 - 10\,(\mathrm{d}x)\,(\mathrm{d}y) + 3\,(\mathrm{d}x)^2 = 0 \Longrightarrow (3\,\mathrm{d}y - \mathrm{d}x)\,(\mathrm{d}y - 3\,\mathrm{d}x) = 0 \tag{26}$$

This leads to

$$\mathrm{d}(3\,y - x) = \mathrm{d}(y - 3\,x) = 0. \tag{27}$$

Therefore the change of variables should be

$$\xi = 3\,y - x, \qquad \eta = y - 3\,x. \tag{28}$$

As this equation is hyperbolic, we know that this change of variables leads to

$$u_{\xi\eta} = \text{lower order terms.} \tag{29}$$

Since the change of variables is linear, and in the original equation there is no lower order term, we conclude that the transformed equation must be

$$u_{\xi\eta} = 0. \tag{30}$$

The general solution is then

$$u(\xi, \eta) = f(\xi) + g(\eta) \tag{31}$$

which translates to

$$u(x, y) = f(3\,y - x) + g(y - 3\,x). \tag{32}$$

Problem 5. (10 pts) Consider the linear equation

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y), \qquad u(x,0) = g(x). \tag{33}$$

with a, b, c, d smooth functions.

a) Introduce a definition for "weak solution" to allow discontinuous solutions.

b) Suppose u is a "weak solution" that is smooth away from a curve Γ. Show that u satisfies the equation in classical sense away from Γ.

c) Suppose u has jump discontinuity along Γ. What is the condition that Γ must satisfy?

Solution. Weak solution

$$\int\int u\,(a\,\phi)_x + u\,(b\,\phi_y) - c\,u\,\phi - d\,\phi\,\mathrm{d}x\,\mathrm{d}y + \int g\,b\,\phi\,\mathrm{d}x = 0. \tag{34}$$

Let Ω_1, Ω_2 be the two domains on both sides of Γ. Gauss' Theorem gives

$$\int_\Gamma \left[u \begin{pmatrix} a \\ b \end{pmatrix} \right] \phi \cdot \boldsymbol{n}\,\mathrm{d}s = 0. \tag{35}$$

As a, b are smooth, this reduces to

$$\int_\Gamma [u]\,\phi \left[\begin{pmatrix} a \\ b \end{pmatrix} \cdot \boldsymbol{n} \right] = 0 \Longrightarrow \begin{pmatrix} a \\ b \end{pmatrix} \cdot \boldsymbol{n} = 0 \tag{36}$$

which means Γ has to be a characteristic curve.