Math 436 Fall 2012 Homework 6 Solutions

Due Dec. 4 in Class

Note. All problem numbers refer to "Updated" version of lecture note.

- **Ex. 3.22.** Construct a sequence $f_n(x) \longrightarrow 0$ for every $x \in \mathbb{R}$, but $\|f_n\| = \left(\int_{\mathbb{R}} f(x)^2 \, \mathrm{d}x\right)^{1/2} = 1$ for all n.

  Solution. Take
  $$f_n(x) = \begin{cases} 1 & x \in (n, n+1) \\ 0 & \text{elsewhere} \end{cases}. \tag{1}$$

- **Ex. 3.23.** Let V be a linear vector space. A norm $\|\cdot\|$ is a mapping $V \mapsto \mathbb{R}$ satisfying

  a) For any $v \in V$, $\|v\| \geqslant 0$, and $\|v\| = 0 \Longleftrightarrow v = 0$.

  b) For any $v \in V$ and $a \in \mathbb{R}$, $\|a\, v\| = |a| \, \|v\|$.

  c) For any $u, v \in V$, $\|u + v\| \leqslant \|u\| + \|v\|$.

  Prove that
  $$\|v\| := \sup_{x \in [a, b]} |f(x)| \tag{2}$$
  is a norm on $V = \{f(x) \colon [a, b] \mapsto \mathbb{R} | \ f(x) \text{ is bounded}\}$. Then show that this norm does not come from an inner product, that is there can be no inner product that $\|v\|^2 = (v, v)$. (Hint: Show that if $(\cdot, \cdot)$ is an inner product, then $(u + v, u + v) + (u - v, u - v) = 2\,(u, u) + 2\,(v, v)$.)

  Proof. a) – c) are quite trivial so omitted.

  To see that the norm does not come from an inner product, we first prove that if $(\cdot, \cdot)$ is an inner product, then $(u + v, u + v) + (u - v, u - v) = 2\,(u, u) + 2\,(v, v)$. To see this, simply use the linearity of inner product:
  $$\begin{aligned} (u+v, u+v) + (u-v, u-v) &= (u, u) + (v, u) + (u, v) + (v, v) \\ &\quad + (u, u) - (v, u) - (u, v) + (v, v) \\ &= 2\,(u, u) + 2\,(v, v). \end{aligned} \tag{3}$$
  Now all we need to do is find $f(x)$, $g(x)$ such that
  $$\|f + g\|^2 + \|f - g\|^2 \neq 2\,(\|f\|^2 + \|g\|^2). \tag{4}$$
  For example we can take
  $$f(x) = \begin{cases} 1 & x \in (0, 1) \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 & x \in (1, 2) \\ 0 & \text{elsewhere} \end{cases}. \tag{5}$$
  □

- **Ex. 4.4.** Consider the linear ODE system

$$\dot{\boldsymbol{x}} = A(t)\,\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{6}$$

Explain why in general

$$\exp\left[\int_0^t A(s)\,\mathrm{d}s\right]\boldsymbol{x}_0 \tag{7}$$

is not a solution.

Solution. Denote $B(t) := \int_0^t A(s)\,\mathrm{d}s$. Then we have $B'(t) = A(t)$. Now calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(e^{B(t)}\,\boldsymbol{x}_0) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}e^{B(t)}\right)\boldsymbol{x}_0 \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(I + B(t) + \frac{B(t)^2}{2} + \cdots\right)\right]\boldsymbol{x}_0 \\ &= \left[B'(t) + \frac{B'(t)\,B(t) + B(t)\,B'(t)}{2} + \cdots\right]\boldsymbol{x}_0 \\ &= \left[A(t) + \frac{A(t)\,B(t) + B(t)\,A(t)}{2} + \cdots\right]\boldsymbol{x}_0.\end{aligned} \tag{8}$$

As $B(t)\,A(t) \neq A(t)\,B(t)$ in general, we cannot write the above as

$$A(t)\,[\cdots] \tag{9}$$

not to say

$$A(t)\,e^{B(t)}\,\boldsymbol{x}_0. \tag{10}$$

- **Ex. 4.5.** Analyze the well/ill-posedness of

$$u_{xx} + u_{tt} + u_t = 0 \tag{11}$$

using normal modes analysis.

Solution. Substitute

$$u = a(k)\,e^{ikx + \lambda(k)t} \tag{12}$$

into the equation, we have

$$-k^2 + \lambda(k)^2 + \lambda(k) = 0 \tag{13}$$

which gives

$$\lambda(k) = \frac{-1 \pm \sqrt{1 + 4\,k^2}}{2}. \tag{14}$$

We have

$$\Omega = \sup_k \Re\lambda(k) = \infty \tag{15}$$

so the problem is ill-posed.