# Math 436 Fall 2012 Homework 5 Solutions

Due Nov. 22 in Class

**Note.** All problem numbers refer to "Updated" version of lecture note.

- **Ex. 3.11.** Write the following equations into S-L form and discuss whether they are regular or singular. Determine what is the orthogonality relation their eigenfunctions should satisfy.

  a) Legendre's equation:

  $$(1-x^2)\, y'' - 2\, x\, y' + \lambda\, y = 0, \qquad -1 < x < 1 \tag{1}$$

  b) Chebyshev's equation

  $$(1-x^2)\, y'' - x\, y' + \lambda\, y = 0, \qquad -1 < x < 1 \tag{2}$$

  c) Laguerre's equation

  $$x\, y'' + (1-x)\, y' + \lambda\, y = 0, \qquad 0 < x < \infty \tag{3}$$

  d) Hermite's equation

  $$y'' - 2\, x\, y' + \lambda\, y = 0, \qquad -\infty < x < \infty \tag{4}$$

  e) Bessel's equation of order $n$

  $$x^2\, y'' + x\, y' + (\lambda\, x^2 - n^2)\, y = 0, \qquad 0 < x < 1. \tag{5}$$

  **Solution.** Can be found everywhere, such as wiki. So omitted.

- **Ex. 3.12.** Give any second order equation

  $$a(x)\, y'' + b(x)\, y' + c\,(x)\, y = 0. \tag{6}$$

  Prove that there exists a multiplier $h(x)$ such that

  $$h(x)\, [a(x)\, y'' + b(x)\, y' + c\,(x)\, y] = (p(x)\, y')' + q(x)\, y. \tag{7}$$

  Note that the term of first order derivative disappears.

  **Proof.** We need

  $$h(x)\, [a(x)\, y'' + b(x)\, y'] = (p(x)\, y')' = p(x)\, y'' + p'(x)\, y'. \tag{8}$$

  Therefore $h$ satisfies

  $$(h(x)\, a(x))' = h(x)\, b(x) \tag{9}$$

  which leads to

  $$h'(x)\, a(x) + h(x)\, a'(x) = h(x)\, b(x) \tag{10}$$

that is

$$\frac{h'(x)}{h(x)} = \frac{b(x)}{a(x)} - \frac{a'(x)}{a(x)}. \tag{11}$$

Therefore

$$\ln h(x) = \int \frac{b(x)}{a(x)}\,\mathrm{d}x - \ln a(x) + C \Longrightarrow h(x) = C\,e^{-a(x)} \exp\left[\int \frac{b(x)}{a(x)}\,\mathrm{d}x\right]. \tag{12}$$

This shows the existence. □

- **Ex. 3.13.** Consider the S-L problem

$$(p\,y')' + q\,y + \lambda\,y = 0, \qquad a < x < b, \qquad y(a) = 0, y(b) = 0. \tag{13}$$

Show that if $p(x) \geqslant 0, q(x) \leqslant M$, then any eigenvalue $\lambda \geqslant -M$.

**Proof.** Let $\lambda$ be an eigenvalue and $y$ be a corresponding eigenfunction. Then we have

$$\begin{aligned} 0 = \int_a^b y\,[(p\,y')' + q\,y + \lambda\,y]\,\mathrm{d}x &= \int_a^b y\,(p\,y')'\,\mathrm{d}x + \int_a^b q\,y^2\,\mathrm{d}x + \lambda \int_a^b y^2\,\mathrm{d}x \\ &\leqslant [y\,(p\,y')]_{x=a}^{x=b} - \int_a^b p\,(y')^2\,\mathrm{d}x \\ &\quad + (M+\lambda) \int_a^b y^2\,\mathrm{d}x \\ &\leqslant (M+\lambda) \int_a^b y^2\,\mathrm{d}x. \end{aligned} \tag{14}$$

As a consequence we must have

$$M + \lambda \geqslant 0 \Longrightarrow \lambda \geqslant -M. \tag{15}$$

Thus ends the proof. □

- **Ex. 3.17.** Prove that the Green's function $G(x, \xi)$ as defined in the notes is symmetric: $G(x, \xi) = G(\xi, x)$. (Hint: Show that $p\,[y_1'\,y_2 - y_1\,y_2']$ is constant).

**Proof.** All we need to show is

$$p(\xi)\,[y_1'(\xi)\,y_2(\xi) - y_1(\xi)\,y_2'(\xi)] \tag{16}$$

is a constant. In other words we need

$$\{p(\xi)\,[y_1'(\xi)\,y_2(\xi) - y_1(\xi)\,y_2'(\xi)]\}' = 0. \tag{17}$$

Simple calculation gives

$$\begin{aligned} \{p(\xi)\,[y_1'(\xi)\,y_2(\xi) - y_1(\xi)\,y_2'(\xi)]\} &= (p\,y_1')'\,y_2 + p\,y_1'\,y_2' \\ &\quad - p\,y_1'\,y_2' - y_1\,(p\,y_2')' \\ &= (-q\,y_1 - \lambda\,y_1)\,y_2 - y_1\,(-q\,y_2 - \lambda\,y_2) \\ &= 0. \end{aligned} \tag{18}$$

Thus ends the proof. □

- **Ex. 3.19.** Let $K$ be defined as

$$Kf := \int_a^b k(x;\xi)\, f(\xi)\,\mathrm{d}\xi \tag{19}$$

where

$$k(x;\xi) = r(x)^{1/2}\, G(x;\xi)\, r(\xi)^{1/2} \tag{20}$$

with $G(x;\xi)$ the Green's function for the operator

$$-(p\,y')' + q\,y \text{ with boundary condition } y(a) = y(b) = 0 \tag{21}$$

in the sense that the solution to

$$-(p\,y')' + q\,y = f, \qquad y(a) = y(b) = 0 \tag{22}$$

is given by

$$y(x) = \int_a^b G(x;\xi)\, f(\xi)\,\mathrm{d}\xi. \tag{23}$$

Assume $p,\ q,\ r > 0$. Show that $K$ is a non-negative operator, that is $(K z,\ z) := \int_a^b [K z]\, z\,\mathrm{d}x \geqslant 0$ for all continuous functions $z$.

**Proof.** First notice that

$$[K z](x) = r(x)^{1/2} \int_a^b G(x;\xi)\, r(\xi)^{1/2}\, z(\xi)\,\mathrm{d}\xi. \tag{24}$$

Therefore if we let $w(x) = r(x)^{-1/2}\, z(x)$, we have

$$r(x)^{-1/2}\, K[z](x) = \int_a^b G(x;\xi)\, r(\xi)\, w(\xi)\,\mathrm{d}\xi \tag{25}$$

which solves the equation

$$-(p\,y')' + q\,y = r\,w. \tag{26}$$

Consequently we have

$$\begin{aligned}(K z, z) &= \int_a^b [K z](x)\, z(x)\,\mathrm{d}x \\ &= \int_a^b [r^{-1/2}(x)\,[K z](x)]\, r(x)\, w(x)\,\mathrm{d}x \\ &= \int_a^b r(x)\, y(x)\, w(x)\,\mathrm{d}x. \end{aligned} \tag{27}$$

As $y$ solves

$$-(p\,y')' + q\,y = r\,w. \tag{28}$$

We conclude

$$\begin{aligned}(K z, z) &= \int_a^b y\,[-(p\,y')' + q\,y]\,\mathrm{d}x \\ &= \int_a^b p\,(y')^2 + q\,y^2\,\mathrm{d}x \geqslant 0 \end{aligned} \tag{29}$$

through integration by parts. □