# Math 436 Fall 2012 Homework 3 Solutions

Due Oct. 25 in Class

**Note.** All problem numbers refer to "Updated" version of lecture note.

- **Ex. 2.28.** d), e). Solve

$$u_t + u_x^2 + u = 0, \quad u(x,0) = x \tag{1}$$

and

$$u_t + u_x^2 = 0, \quad u(x,0) = -x^2. \tag{2}$$

Show that the solution of the latter breaks down when $t = 1/4$.
**Solution.**

  - d). We have $F(x,t,z,p,q) = q + p^2 + z$. The system of characteristics is

$$\begin{aligned} \dot{x} &= F_p = 2\,p & (3)\\ \dot{t} &= F_q = 1 & (4)\\ \dot{z} &= p\,F_p + q\,F_q = 2\,p^2 + q = p^2 - z & (5)\\ \dot{p} &= -F_x - p\,F_z = p & (6)\\ \dot{q} &= -F_t - q\,F_z = q & (7) \end{aligned}$$

with intial conditions

$$x_0 = \tau, \qquad t_0 = 0, \qquad z_0 = \tau \tag{8}$$

and $p_0$, $q_0$ satisfy

$$q_0 + p_0^2 + z_0 = 0, \qquad 1 = \frac{\mathrm{d}z_0}{\mathrm{d}\tau} = p_0 \frac{\mathrm{d}x_0}{\mathrm{d}\tau} + q_0 \frac{\mathrm{d}t_0}{\mathrm{d}\tau} = p_0. \tag{9}$$

This leads to

$$p_0 = 1, \qquad q_0 = -\tau - 1. \tag{10}$$

From this we solve

$$p(\tau,s) = e^s, \qquad q(\tau,s) = -(\tau+1)\,e^s. \tag{11}$$

Substituting into other equations we obtain

$$x(\tau,s) = \tau + 2\,e^s, \qquad t(\tau,s) = s, \qquad z(\tau,s) = \frac{e^{3s}}{3} + \tau\,e^{-s}. \tag{12}$$

From the $x,t$ equations we have $\tau = x - 2\,e^t$. Thus

$$u(x,t) = z(\tau,s) = \frac{e^{3t}}{3} + (x - 2\,e^t)\,e^{-t} = \frac{e^{3t}}{3} + x\,e^{-t} - 2. \tag{13}$$

  - e). We have $F(x,t,z,p,q) = q + p^2$. The system of characteristics is

$$\begin{aligned} \dot{x} &= F_p = 2\,p & (14)\\ \dot{t} &= F_q = 1 & (15)\\ \dot{z} &= p\,F_p + q\,F_q = 2\,p^2 + q = p^2 & (16)\\ \dot{p} &= -F_x - p\,F_z = 0 & (17)\\ \dot{q} &= -F_t - q\,F_z = 0 & (18) \end{aligned}$$

with intial conditions

$$x_0 = \tau, \qquad t_0 = 0, \qquad z_0 = -\tau^2 \tag{19}$$

and

$$q_0 + p_0^2 = 0, \qquad -2\,\tau = \frac{\mathrm{d}z_0}{\mathrm{d}\tau} = p_0\,\frac{\mathrm{d}x_0}{\mathrm{d}\tau} + q_0\,\frac{\mathrm{d}t_0}{\mathrm{d}\tau} = p_0. \tag{20}$$

This gives

$$p(\tau, s) = p_0 = -2\,\tau, \qquad q(\tau, s) = q_0 = -4\,\tau^2. \tag{21}$$

Consequently we have

$$x(\tau, s) = \tau - 4\,\tau\,s, \qquad t(\tau, s) = s, \qquad z(\tau, s) = 4\,\tau^2\,s - \tau^2 = (4\,s - 1)\,\tau^2. \tag{22}$$

The $x, t$ equations now give

$$s = t, \qquad \tau = \frac{x}{1 - 4\,t} \tag{23}$$

and consequently

$$u(x, t) = z(\tau, s) = \frac{x^2}{4\,t - 1}. \tag{24}$$

It is clear that the solution breaks down at $t = 1/4$.

- **Ex. 2.29. (Snell's law)** Consider the eiconal equation

$$u_x^2 + u_y^2 = n(x, y)^2, \qquad n(x, y) = \begin{cases} n_1 & y < 0 \\ n_2 & y > 0 \end{cases}. \tag{25}$$

Here $n_2 > n_1$ are constants. Let the initial condition be $u(x,\,0) = n_1\,x\,\cos\theta$ with $\theta \in \left[0, \frac{\pi}{2}\right]$.

a) Solve the equation.

b) By considering the directions $\nabla u$, confirm Snell's law.[1]

**Solution.**

a) We have $F(x, y, z, p, q) = p^2 + q^2 - n(x, y)^2$. Thus the system for characteristics is

$$\begin{aligned} \dot{x} &= F_p = 2\,p, & (26)\\ \dot{y} &= F_q = 2\,q, & (27)\\ \dot{z} &= p\,F_p + q\,F_q = 2\,(p^2 + q^2) = 2\,n(x, y)^2 = \begin{cases} 2\,n_1^2 & y < 0 \\ 2\,n_2^2 & y > 0 \end{cases}, & (28)\\ \dot{p} &= -F_x - p\,F_z = 0, & (29)\\ \dot{q} &= -F_y - q\,F_z = 0. & (30) \end{aligned}$$

with initial conditions

$$x_0 = \tau, \, y_0 = 0, \, u_0 = n_1\,\tau\,\cos\theta. \tag{31}$$

1. Check wiki if you forget what it is.

We determine $p_0, q_0$ from

$$p_0^2+q_0^2=\begin{cases} n_1^2 & y=0-\\ n_2^2 & y=0+\end{cases}, \qquad n_1\cos\theta=\frac{\mathrm{d}u_0}{\mathrm{d}\tau}=p_0\frac{\mathrm{d}x_0}{\mathrm{d}\tau}+q_0\frac{\mathrm{d}y_0}{\mathrm{d}\tau}=p_0. \tag{32}$$

Therefore we have $p_0=n_1\cos\theta$, $q_0=\begin{cases}\pm n_1\sin\theta & y=0-\\ \pm\sqrt{n_2^2-(n_1\cos\theta)^2} & y=0+\end{cases}$. Now notice that $\dot{y}=2\,q$. If we assume that when $s>0$, $y>0$, we must have $\dot{y}>0$ at $y=0$ therefore

$$q_0=\begin{cases} n_1\sin\theta & y=0-\\ \sqrt{n_2^2-(n_1\cos\theta)^2} & y=0+\end{cases}. \tag{33}$$

This leads to

$$p(\tau,s)=n_1\cos\theta, \qquad q(\tau,s)=\begin{cases} n_1\sin\theta & y<0\\ \sqrt{n_2^2-(n_1\cos\theta)^2} & y>0\end{cases}. \tag{34}$$

Next we solve the $x,y,z$ equations:

$$\dot{x}=2\,p\Longrightarrow x=\tau+2\,n_1\,s\cos\theta, \tag{35}$$

$$\dot{y}=2\,q\Longrightarrow y=\begin{cases} 2\,n_1\,s\sin\theta & s<0\\ 2\,s\sqrt{n_2^2-(n_1\cos\theta)^2} & s>0\end{cases}, \tag{36}$$

$$\dot{z}=\begin{cases} 2\,n_1^2 & y<0\\ 2\,n_2^2 & y>0\end{cases}\Longrightarrow z(s,\tau)=\begin{cases} n_1\,\tau\cos\theta+2\,n_1^2\,s & y<0\\ n_1\,\tau\cos\theta+2\,n_2^2\,s & y>0\end{cases}. \tag{37}$$

Thus we have

$$s=\begin{cases} \dfrac{y}{2\,n_1\sin\theta} & y<0\\ \dfrac{y}{2\sqrt{n_2^2-(n_1\cos\theta)^2}} & y>0\end{cases}, \tau=\begin{cases} x-y\dfrac{\cos\theta}{\sin\theta} & y<0\\ x-\dfrac{y\,n_1\cos\theta}{\sqrt{n_2^2-(n_1\cos\theta)^2}} & y>0\end{cases} \tag{38}$$

which leads to

$$u(x,y)=\begin{cases} n_1\,[x\cos\theta+y\sin\theta] & y<0\\ n_1\,x\cos\theta+y\sqrt{n_2^2-(n_1\cos\theta)^2} & y>0\end{cases}. \tag{39}$$

b) We have

$$u_x=n_1\cos\theta, \qquad u_y=\begin{cases} n_1\sin\theta & y<0\\ \sqrt{n_2^2-(n_1\cos\theta)^2} & y>0\end{cases}. \tag{40}$$

Here $\theta$ is the angle between the vector $\nabla u$ and the interface $y=0$ in $y<0$. If we let $\tilde{\theta}$ be the angle between $\nabla u$ and $y=0$ in $y>0$, we have

$$\cos\tilde{\theta}=\frac{u_x}{(u_x^2+u_y^2)^{1/2}}=\frac{n_1}{n_2}\cos\theta. \tag{41}$$

This is exactly Snell's law.

- **Ex. 2.32.** a) Reduce the following equation

$$u_{xx} + 4\,u_{xy} + 3\,u_{yy} + 3\,u_x - u_y + u = 0 \tag{42}$$

to canonical form. Then use further transformation

$$u(\xi, \eta) = \exp(\alpha\,\xi + \beta\,\eta)\,v(\xi, \eta) \tag{43}$$

and choose the constants $\alpha, \beta$ to eliminate the first derivative terms.
**Solution.** We have

$$(\mathrm{d}y)^2 - 4\,(\mathrm{d}x)\,(\mathrm{d}y) + 3\,(\mathrm{d}x)^2 = 0 \Longrightarrow (\mathrm{d}y - 3\,\mathrm{d}x)(\mathrm{d}y - \mathrm{d}x) = 0 \tag{44}$$

so we set $\xi = y - 3\,x,\ \eta = y - x$. So $\xi_x = -3, \xi_y = 1, \eta_x = -1, \eta_y = 1$. This leads to

$$u_x = -3\,u_\xi - u_\eta, \qquad u_y = u_\xi + u_\eta \tag{45}$$

and

$$u_{xx} = 9\,u_{\xi\xi} + 6\,u_{\xi\eta} + u_{\eta\eta}, u_{xy} = -3\,u_{\xi\xi} - 4\,u_{\xi\eta} - u_{\eta\eta}, u_{yy} = u_{\xi\xi} + 2\,u_{\xi\eta} + u_{\eta\eta} \tag{46}$$

Thus the equation in $\xi$-$\eta$ variables reads

$$4\,u_{\xi\eta} + 10\,u_\xi + 4\,u_\eta - u = 0. \tag{47}$$

If we introduce $v(\xi, \eta)$ through

$$u(\xi, \eta) = \exp(\alpha\,\xi + \beta\,\eta)\,v(\xi, \eta) \tag{48}$$

then we have

$$u_\xi = \alpha\,e^{\alpha\xi+\beta\eta}\,v + e^{\alpha\xi+\beta\eta}\,v_\xi, \qquad u_\eta = \beta\,e^{\alpha\xi+\beta\eta}\,v + e^{\alpha\xi+\beta\eta}\,v_\eta. \tag{49}$$

and

$$u_{\xi\eta} = e^{\alpha\xi+\beta\eta}\,v_{\xi\eta} + \alpha\,e^{\alpha\xi+\beta\eta}\,v_\eta + \beta\,e^{\alpha\xi+\beta\eta}\,v_\xi + \alpha\,\beta\,e^{\alpha\xi+\beta\eta}\,v. \tag{50}$$

Substituting into the equation we obtain

$$e^{\alpha\xi+\beta\eta}\,[4\,v_{\xi\eta} + (4\,\beta + 10)\,v_\xi + (4\,\alpha + 4)\,v_\eta + (4\,\alpha\,\beta + 10\,\alpha + 4\,\beta - 1)\,v] = 0. \tag{51}$$

Choosing $\alpha = -5/2, \beta = -1$ we reach

$$v_{\xi\eta} - 5\,v = 0. \tag{52}$$

- **Ex. 2.33.** Consider the general linear 2nd order equation in $\mathbb{R}^n$:

$$\sum_{i,j}^{n} a_{ij}\,u_{x_i x_j} + \sum_{i=1}^{n} b_i\,u_{x_i} + c\,u + d = 0. \tag{53}$$

with constant coefficients. Prove that there is a change of variables which reduce the equation to canonical form.

**Proof.** Define the matrix $A = (a_{ij})$. From linear algebra we know that there is a nonsingular matrix $R$ such that

$$R\,A\,R^T = \begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0_{n-p-q} \end{pmatrix} \tag{54}$$

where $I_p$ means the $p\times p$ identity matrix, $0_r$ means the $r\times r$ zero matrix.

Since $a_{ij}$ are constants, $R$ is a constant matrix. Denoting its $(i,j)$ entry by $r_{ij}$, we introduce the change of variables:

$$\xi_i = r_{i1}\,x_1 + r_{i2}\,x_2 + \cdots + r_{in}\,x_n. \tag{55}$$

Then straightforward calculation shows that under such change of variables, the equation becomes

$$\sum_{i,j=1}^{n} \tilde{a}_{ij}\,u_{\xi_i\xi_j} + \text{lower order terms}=0 \tag{56}$$

with the matrix

$$\tilde{A} = (\ \tilde{a}_{ij}\ ) = \begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0_{n-p-q} \end{pmatrix}. \tag{57}$$

This means the equation reads

$$u_{\xi_1\xi_1} + \cdots + u_{\xi_p\xi_p} - u_{\xi_{p+1}\xi_{p+1}} - \cdots - u_{\xi_{p+q}\xi_{p+q}} + \text{lower order terms}=0. \tag{58}$$

which is exactly the canonical form. □

- **Ex. 2.40.** Let $\varphi(x,y) = \text{constant}$ be a family of characteristics for

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y). \tag{59}$$

Let $\xi = \varphi(x,y)$ and $\eta = \psi(x,y)$ be perpendicular to it. Show that the equation reduces to

$$(a\,\psi_x + b\,\psi_y)\,u_\eta = c\,u + d. \tag{60}$$

Now assume $u$ is continuous across $\xi = 0$ while $u_\xi$ has a jump there, then the jump $[u_\xi]$ satisfies

$$(a\,\psi_x + b\,\psi_y)[u_\xi]_\eta = c\,[u_\xi]. \tag{61}$$

Thus the propagation of jumps are determined by a equation.

**Proof.** Since $\varphi(x,y) = c$ is characteristics, we have $a\,\varphi_x + b\,\varphi_y = 0$. Apply chain rule we have

$$a\,u_x + b\,u_y = (a\,\varphi_x + b\,\varphi_y)\,u_\xi + (a\,\psi_x + b\,\psi_y)\,u_\eta = (a\,\psi_x + b\,\psi_y)\,u_\eta \tag{62}$$

That is the equation reduces to

$$(a\,\psi_x + b\,\psi_y)\,u_\eta = c\,u + d. \tag{63}$$

If we further assume $u$ is continuous across $\xi = 0$ while $u_\xi$ has a jump there, then we can take $\partial_\xi$ of both sides of the equation to obtain

$$(a\,\psi_x + b\,\psi_y)\,u_{\xi\eta} + (a\,\psi_x + b\,\psi_y)_\xi\,u_\eta = c\,u_\xi + c_\xi\,u + d_\xi. \tag{64}$$

As $a, b, c, d$ are functions of $x, y$ (that is $\xi, \eta$) only, all the terms in the above equations are continuous across $\xi = 0$ except $(a\,\psi_x + b\,\psi_y)\,u_{\xi\eta}$ and $c\,u_\xi$. Taking the difference between $\xi = 0+$ and $0-$ gives the desired result. □

- **Ex. 2.42.** Consider the first order quasi-linear equation:

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u). \tag{65}$$

Assume that $u$ is smooth everywhere except that along $\Phi = 0$ there is a "jump" in its 2nd order derivatives. Derive the equation for $\Phi$. Then consider the case when the "jump" is in its $k$th derivative and all $(k-1)$th derivatives are continuous across the curve.

**Proof.** We consider directly the general case. We do a change of variables $\xi = \Phi(x, y)$ and $\eta = \Psi(x, y)$ with level sets of $\Psi$ perpendicular to that of $\Phi$. Then the equation becomes

$$(a\,\Phi_x + b\,\Phi_y)\, u_\xi + (a\,\Psi_x + b\,\Psi_y)\, u_\eta = c. \tag{66}$$

Since all the $(k-1)$th derivatives are continuous across the curve, we have all the $k$th derivatives are also continuous except for $\partial_\xi^k u$: For any $l \geqslant 1$, we can write

$$\partial_\eta^l \partial_\xi^{k-l} u = \partial_\eta(\partial_\eta^{l-1} \partial_\xi^{k-l} u) \tag{67}$$

which is a $\eta$ derivative of a $(k-1)$th order derivative. Therefore taking $\partial_\xi^{k-1}$ of (66) we reach

$$(a\,\Phi_x + b\,\Phi_y)\,(\partial_\xi^k u) + [\text{terms continuous across } \xi = 0] = 0. \tag{68}$$

Now taking the difference between $\xi = 0+$ and $0-$ we conclude

$$(a\,\Phi_x + b\,\Phi_y)\,[\partial_\xi^k u] = 0 \Longrightarrow a\,\Phi_x + b\,\Phi_y \tag{69}$$

which is the equation for characteristic curves. □