Math 436 Fall 2012 Homework 3

Due Oct. 25 in Class

Note. All problem numbers refer to "Updated" version of lecture note.

- **Ex. 2.28.** d), e). Solve
$$u_t + u_x^2 + u = 0, \quad u(x,0) = x \tag{1}$$
and
$$u_t + u_x^2 = 0, \quad u(x,0) = -x^2. \tag{2}$$
Show that the solution of the latter breaks down when $t = 1/4$.

- **Ex. 2.29. (Snell's law)** Consider the eiconal equation
$$u_x^2 + u_y^2 = n(x,y)^2, \qquad n(x,y) = \begin{cases} n_1 & y < 0 \\ n_2 & y > 0 \end{cases}. \tag{3}$$
Here $n_2 > n_1$ are constants. Let the initial condition be $u(x, 0) = n_1\, x \cos\theta$ with $\theta \in \left[0, \frac{\pi}{2}\right]$.

    a) Solve the equation.

    b) By considering the directions ∇u, confirm Snell's law.[1]

- **Ex. 2.32.** a) Reduce the following equation
$$u_{xx} + 4\, u_{xy} + 3\, u_{yy} + 3\, u_x - u_y + u = 0 \tag{4}$$
to canonical form. Then use further transformation
$$u(\xi, \eta) = \exp(\alpha\, \xi + \beta\, \eta)\, v(\xi, \eta) \tag{5}$$
and choose the constants α, β to eliminate the first derivative terms.

- **Ex. 2.33.** Consider the general linear 2nd order equation in $\mathbb{R}^n$:
$$\sum_{i,j}^{n} a_{ij}\, u_{x_i x_j} + \sum_{i=1}^{n} b_i\, u_{x_i} + c\, u + d = 0. \tag{6}$$
with constant coefficients. Prove that there is a change of variables which reduce the equation to canonical form.

- **Ex. 2.40.** Let $\varphi(x,y) =$ constant be a family of characteristics for
$$a(x,y)\, u_x + b(x,y)\, u_y = c(x,y)\, u + d(x,y). \tag{7}$$
Let $\xi = \varphi(x,y)$ and $\eta = \psi(x,y)$ be perpendicular to it. Show that the equation reduces to
$$(a\, \psi_x + b\, \psi_y)\, u_\eta = c\, u + d. \tag{8}$$

1. Check wiki if you forget what it is.

Now assume u is continuous across $\xi = 0$ while u_ξ has a jump there, then the jump $[u_\xi]$ satisfies

$$(a\,\psi_x + b\,\psi_y)[u_\xi]_\eta = c\,[u_\xi]. \tag{9}$$

Thus the propagation of jumps are determined by a equation.

- **Ex. 2.42.** Consider the first order quasi-linear equation:

$$a(x, y, u)\,u_x + b(x, y, u)\,u_y = c(x, y, u). \tag{10}$$

Assume that u is smooth everywhere except that along $\Phi = 0$ there is a "jump" in its 2nd order derivatives. Derive the equation for Φ. Then consider the case when the "jump" is in its kth derivative and all $(k-1)$th derivatives are continuous across the curve.