Math 436 Fall 2012 Homework 2

Due Oct. 11 in Class

Note. All problem numbers refer to "Updated" version of lecture note.

- **Ex. 2.2.** Find the solution of the following Cauchy problems.

  a) $x\,u_x + y\,u_y = 2\,x\,y, \qquad$ with $u = 2$ on $y = x^2$.

  b) $u\,u_x - u\,u_y = u^2 + (x+y)^2 \qquad$ with $u = 1$ on $y = 0$.

- **Ex. 2.4.** Consider a quasi-linear equation

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u) \tag{1}$$

  (without specifying any initial conditions). Let $u_1(x,\,y)$, $u_2(x,\,y)$ be two solutions. Assume that the surfaces $u - u_1(x,y) = 0$ and $u - u_2(x,y) = 0$ intersects along a curve Γ in the xyu space. Show that Γ must be a characteristic curve.

- **Ex. 2.7.** Show that the initial value problem

$$u_t + u_x = 0, \qquad u = x \text{ on } x^2 + t^2 = 1 \tag{2}$$

  has no solution. However, if the initial data are given only over the semicircle that lies in the half-plane $x + t \leqslant 0$, the solution exists but is not differentiable along the characteristic base curves that issue from the two end points of the semicircle.

- **Ex. 2.12.** Consider the wave equation

$$u_{tt} - u_{xx} = 0, \qquad u(x,0) = g(x), \quad u_t(x,0) = h(x). \tag{3}$$

  Show that

  a) If we set $v(x,t) = u_t - u_x$, then v satisfies

$$v_t + v_x = 0, \qquad v(x,0) = h(x) - g'(x). \tag{4}$$

  b) Use method of characteristics to solve the v equation and then the u equation. Show that the solution is given by the d'Alembert's formula

$$u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(s)\,\mathrm{d}s. \tag{5}$$

- **Ex. 2.18.** Solve (that is construct entropy solution for all t)

$$u_t + \left(\frac{u^4}{4}\right)_x = 0, \qquad u(0,x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}. \tag{6}$$

- **Ex. 2.19.** Compute explicitly the unique entropy solution of

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \qquad u(0,x) = g \tag{7}$$

for

$$g(x) = \begin{cases} 1 & x < -1 \\ 0 & -1 < x < 0 \\ 2 & 0 < x < 1 \\ 0 & x > 1 \end{cases}. \tag{8}$$

Draw a picture of your answer. Be sure to illustrate what happens for all times $t > 0$.

- **Ex. 2.22.** Prove that

$$u(t,x) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < t \\ 1 & x > t \end{cases} \tag{9}$$

is a weak solution to the problem

$$u_t + u\,u_x = 0, \qquad u(0,x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}. \tag{10}$$