Math 317 Winter 2014 Complimentary Quiz (Apr. 21, 2014)

Warning: This is not a sample exam.

(D) : Difficult; (C) : Challenge.

QUESTION 1. *Study the convergence, continuity, and differentiability of*

$$\sum_{n=1}^{\infty} \frac{\sin\left(\left(\frac{n+1}{n}\right)^n x\right)}{(n+1)\,n}. \tag{1}$$

Solution. We have

$$\left|\frac{\sin\left(\left(\frac{n+1}{n}\right)^n x\right)}{(n+1)\,n}\right| \leqslant \frac{1}{(n+1)\,n} \tag{2}$$

and

$$\left|\left[\frac{\sin\left(\left(\frac{n+1}{n}\right)^n x\right)}{(n+1)\,n}\right]'\right| \leqslant \left(\frac{n+1}{n}\right)^n \frac{1}{(n+1)\,n} < \frac{e}{(n+1)\,n}. \tag{3}$$

Thus convergence, continuity and differentiability at all x follow from the M-test.

QUESTION 2. *Let $f(x)$ be 2π periodic and equals $x+1$ on $[-\pi,\pi]$. Find its Fourier expansion and determine the function to which the Fourier series converge to.*

Solution. We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+1)\,\mathrm{d}x = 2. \tag{4}$$

We can see that $a_n = 0$ and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+1)\sin(n\,x)\,\mathrm{d}x \\ &= \frac{1}{-n\,\pi}\int_{-\pi}^{\pi} (x+1)\,\mathrm{d}\cos(n\,x) \\ &= -\frac{1}{n\,\pi}\left[(x+1)\cos(n\,x)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\cos(n\,x)\,\mathrm{d}x\right] \\ &= -\frac{1}{n\,\pi}\left[(\pi+1)(-1)^n - (-\pi+1)(-1)^n\right] \\ &= \frac{2(-1)^{n+1}}{n}. \end{aligned} \tag{5}$$

The Fourier series converges to $x+1$ on $(-\pi,\pi)$ and 1 at $\pm\pi$.

QUESTION 3. *Let $A := \{$ellipsoids in $\mathbb{R}^3\}$. Find its cardinality.*

Solution. An ellipsoid is determined by: center $\in \mathbb{R}^3$, 3 axes each $\in \mathbb{R}^3$. So we have $A \lesssim \mathbb{R}^{12} \sim \mathbb{R}$. On the other hand consider unit spheres centered along the x-axis, we have $A \gtrsim \mathbb{R}$. Therefore $A \sim \mathbb{R}$.

QUESTION 4. *Well-order $\mathbb{N}\times\mathbb{N}$. What is the ordinal number of your re-ordered set?*

QUESTION 5. *Calculate the surface area of S: $\{(x,y,z) \mid x^2+y^2+z^2=3, z^2 \geqslant 2x^2+2y^2\}$.*

Solution. First note that S has two parts. Its area is two times that of

$$S_u := \{(x,y,z) \mid x^2+y^2+z^2=3, z \geqslant 0, z^2 \geqslant 2x^2+2y^2\}. \tag{6}$$

We parametrize S_u as follows: First $S=\{(x,\, y,\, z)|\, x^2+y^2+z^2=3,\, x^2+y^2\leqslant 1\}$. Thus we can take the parametrization: $\begin{pmatrix} u \\ v \\ \sqrt{3-u^2-v^2} \end{pmatrix}$, $D=\{(u,v)|\, u^2+v^2\leqslant 1\}$. We calculate

$$\boldsymbol{r}_u=\begin{pmatrix} 1 \\ 0 \\ -\frac{u}{\sqrt{3-u^2-v^2}} \end{pmatrix}, \qquad \boldsymbol{r}_v=\begin{pmatrix} 0 \\ 1 \\ -\frac{v}{\sqrt{3-u^2-v^2}} \end{pmatrix}. \tag{7}$$

This gives

$$E=1+\frac{u^2}{3-u^2-v^2}, \quad F=\frac{u\,v}{3-u^2-v^2}, \quad G=1+\frac{v^2}{3-u^2-v^2} \tag{8}$$

and

$$EG-F^2=1+\frac{u^2+v^2}{3-u^2-v^2}=\frac{3}{3-u^2-v^2}. \tag{9}$$

(Alternatively, since the surface is given by $z=\phi(x,y)$ where $\phi(x,y):=\sqrt{3-x^2-y^2}$, we have $\text{Area}(S_u)=\int_D \sqrt{1+\phi_x^2+\phi_y^2}\,\text{d}(x,y)$)

Thus

$$\begin{aligned}
\text{Area}(S_u) &= \int_D \sqrt{EG-F^2}\,\text{d}(u,v) \\
&= \int_{u^2+v^2\leqslant 1} \sqrt{\frac{3}{3-u^2-v^2}}\,\text{d}(u,v) \\
&= 2\pi \int_0^1 \sqrt{\frac{3}{3-r^2}}\, r\,\text{d}r \\
&= \sqrt{3}\,\pi \int_0^1 \frac{1}{\sqrt{3-u}}\,\text{d}u \\
&= \sqrt{3}\,\pi[-2\sqrt{3-u}]_0^1 \\
&= 2\sqrt{3}\,\pi\left[\sqrt{3}-\sqrt{2}\right].
\end{aligned} \tag{10}$$

The area of S is then $4\sqrt{3}\,\pi\left[\sqrt{3}-\sqrt{2}\right]$.

QUESTION 6. *Calculate*

$$\int_S \begin{pmatrix} 1 \\ 2 \\ z \end{pmatrix} \cdot \mathbf{d}\boldsymbol{S} \tag{11}$$

where $S:=\{(x,y,z)|\, x^2+y^2+z^2=2, z\geqslant x^2+y^2\}$ *with normal pointing upward*

i. directly;

ii. (D) *using Gauss's Theorem;*

iii. (C) *Can you calculate the integral using Stokes's Theorem? Explain.*

Solution.

i. $z=x^2+y^2$ at $x^2+y^2=1$. Therefore $S=\left\{z=\sqrt{2-x^2-y^2}, x^2+y^2\leqslant 1\right\}$. We have

$$\boldsymbol{n}\,\text{d}S=\begin{pmatrix} -z_x \\ -z_y \\ 1 \end{pmatrix}\text{d}(x,y)=\begin{pmatrix} \frac{x}{\sqrt{2-x^2-y^2}} \\ \frac{y}{\sqrt{2-x^2-y^2}} \\ 1 \end{pmatrix}\text{d}(x,y). \tag{12}$$

Thus we integrate

$$I = \int_{x^2+y^2 \leqslant 1} \sqrt{2-x^2-y^2}\, \mathrm{d}(x,y) = 2\pi \int_0^1 \sqrt{2-r^2}\, r\, \mathrm{d}r = \frac{2}{3}\left(2^{3/2}-1\right). \tag{13}$$

ii. Gauss: Take $V := \{x^2+y^2+z^2 \leqslant 2, z \geqslant 1, x^2+y^2 \leqslant 1\}$. Then we have $\partial V = S \cup S_{\text{bottom}}$. We have

$$\int_V \mathrm{d}\boldsymbol{x} = \int_S + \int_{S_{\text{bottom}}}. \tag{14}$$

Now we calculate

$$\begin{aligned}\int_V \mathrm{d}\boldsymbol{x} &= \int_{x^2+y^2 \leqslant 1} \left[\int_1^{\sqrt{2-x^2-y^2}} \mathrm{d}z\right] \mathrm{d}(x,y) \\ &= \int_{x^2+y^2 \leqslant 1} \left[\sqrt{2-x^2-y^2} - 1\right] \mathrm{d}(x,y) = \frac{2}{3}\left(2^{3/2}-1\right) - \pi.\end{aligned}$$

As S_{bottom} is the disc $\{(x,y,z) | z = \phi(x,y) = 1, x^2+y^2 \leqslant 1\}$ with normal pointing downward, we have

$$\int_{S_{\text{bottom}}} \begin{pmatrix} 1 \\ 2 \\ z \end{pmatrix} \cdot \mathbf{d}\boldsymbol{S} = \int_{x^2+y^2 \leqslant 1} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -\pi. \tag{15}$$

Thus

$$\int_S \begin{pmatrix} 1 \\ 2 \\ z \end{pmatrix} \cdot \mathbf{d}\boldsymbol{S} = \frac{2}{3}\left(2^{3/2}-1\right). \tag{16}$$

iii. No. Because if $\begin{pmatrix} 1 \\ 2 \\ z \end{pmatrix} = \nabla \times \boldsymbol{f}$, then we must have $\operatorname{div}\begin{pmatrix} 1 \\ 2 \\ z \end{pmatrix} = 0$ which is not satisfied. Or more directly, we need to solve

$$h_y - g_z = 1, \quad f_z - h_x = 2, \quad g_x - f_y = z. \tag{17}$$

Taking $\frac{\partial}{\partial x}$ of the first equation and $\frac{\partial}{\partial y}$ of the second equation we have

$$g_{xz} = h_{xy} = f_{yz}. \tag{18}$$

But taking $\frac{\partial}{\partial z}$ of the 3rd equation we have $g_{xz} - f_{yz} = 1$. Thus it is not possible to find $\boldsymbol{f}$ such that $\begin{pmatrix} 1 \\ 2 \\ z \end{pmatrix} = \nabla \times \boldsymbol{f}$.

QUESTION 7. *Consider the infinite series of functions*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\cos x)^n. \tag{19}$$

a) *Find all x that the series is convergent.*

b) (D) *Denote the sum by $f(x)$. Discuss its continuity.*

c) (C) *Discuss its differentiability.*

Exercise 1. Prove the following:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} u^n \text{ converges/diverges at } u = \cos x \Leftrightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\cos x)^n \text{ converges/diverges at } x;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} u^n \text{ converges uniformly on } [a,b] \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\cos x)^n \text{ converges uniformly on } A$$

where $A := \{x \in \mathbb{R} | \cos x \in [a, b]\}$.

Solution.

a) We know that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} u^n$ converges for all $|u| \leqslant 1$ except $u = -1$. Thus $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\cos x)^n$ converges for all $x \in \mathbb{R}$ except $x = (2k+1)\pi$ for $k \in \mathbb{Z}$.

b) From Abel's Theorem we know that the convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} u^n$ is uniform on $(-1+\varepsilon, 1]$ for all $\varepsilon > 0$. Thus the convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\cos x)^n$ is uniform on $\cup_{k\in\mathbb{Z}}((2k-1)\pi + \varepsilon, (2k+1)\pi - \varepsilon)$ for every $\varepsilon > 0$. So $f(x)$ is continuous on its domain.

c) Take derivative termwise:

$$\sum_{n=1}^{\infty} (-1)^{n+1} (\cos x)^{n-1} \sin x = -\frac{\sin x}{\cos x} \sum_{n=1}^{\infty} (-\cos x)^n. \tag{20}$$

so uniform convergence on $\delta < |x| < \pi - \delta$ is obvious. Also it obviously converges at $x = 0$. For $0 < |x| < \delta$ we have

$$\sin x \sum_{n=N}^{\infty} (-\cos x)^{n-1} = \sin x \, (-\cos x)^{N-1} \frac{1}{1+\cos x} \tag{21}$$

whose absolute value is bounded by $|\sin x|$.

We have for any $\varepsilon > 0$, take $\delta > 0$ such that $|\sin \delta| < \varepsilon$. Then we see that for all $0 < |x| < \delta$,

$$\left| \sum_{n=N}^{\infty} (-\cos x)^{n-1} \sin x \right| = |\sin x| \, |\cos x|^{N-1} \frac{1}{1+\cos x} < \varepsilon. \tag{22}$$

Thus the convergence is uniform and f is differentiable everywhere it is defined.

QUESTION 8. (A) *Let* $A := \{f : [0,1) \mapsto \mathbb{R} | \, f$ *is a piecewise constant function.*$\}$ *where* f *is a piecewise constant function if and only if there is a partition* $\{0 = x_0 < x_1 < \cdots < x_n = 1\}$ *such that* f *is constant on each* $[x_i, x_{i+1})$.

Solution. For each fixed $0 = x_0 < \cdots < x_n = 1$, we have the number of functions $\mathbb{R}^n \sim \mathbb{R}$. Now there are $\mathbb{R}^{n-1}$ possibilities of $(x_1, \ldots, x_{n-1})$ so the number of functions for each n is no more than $\mathbb{R}^n \sim \mathbb{R}$. Finally take union over n we have no more than $\mathbb{N} \cdot \mathbb{R} \sim \mathbb{R}$. Obviously $A \gtrsim \mathbb{R}$. By Schröder-Bernstein we have $A \sim \mathbb{R}$.

Alternatively, each f is obtained through finitely many times of the following three operations: multiply by $a \in \mathbb{R}$, translate by $b \in \mathbb{R}$, "flip" horizontally, on the step function.