NAME: ID:

Math 317 Quiz 5 Solutions

MAR. 17, 2014

- The quiz has three problems. Total $10+1$ points. It should be completed in 20 minutes.

Question 1. (5 pts) *Let A be the set of all finite rectangles in $\mathbb{R}^2$. Find its cardinality. Justify your answer.*

Solution. First for any $x\in\mathbb{R}$, $[0,x]^2\in A$. Therefore $A\gtrsim\mathbb{R}$; On the other hand we have the following injection $f\colon A\mapsto\mathbb{R}^8$:

$$R\mapsto(a_1,b_1,a_2,b_2,a_3,b_3,a_4,b_4) \tag{1}$$

where (a_i,b_i) are coordinates of the vertices.

Therefore $A\lesssim\mathbb{R}^8$. But $\mathbb{R}^8\sim\mathbb{R}$. Thus by Schroeder-Bernstein $A\sim\mathbb{R}$, that is the cardinality is $\mathfrak{c}$.

Remark. I should have specified that the rectangles should be compact. But looks like nobody's solution was actually affected by this anyway.

Question 2. (5 pts) *Re-order $\mathbb{N}$ to have ordinal number $\omega^2+\omega\cdot 2+1$.*

Solution. For $k\in\mathbb{N}$, let $A_k:=\{n\in\mathbb{N}|\, n$ has exactly k prime factors$\}$ ordered by the natural order. Then we have

$$\mathbb{N}=\{1\}\cup_{k=1}^{\infty}A_k \tag{2}$$

and we can re-order $\mathbb{N}$ as

$$A_3<A_4<\cdots<A_1<A_2<1. \tag{3}$$

Question 3. (1 bonus pt) *Prove the following theorem due to Paul du Bois-Reymond (1831 - 1889):*

Given any sequence of functions $f_m\colon\mathbb{N}\mapsto\mathbb{N}$, $m=1,2,3,\ldots.$, there is a function $f\colon\mathbb{N}\mapsto\mathbb{N}$ such that

$$\forall m\in\mathbb{N},\qquad \lim_{n\to\infty}\frac{f_m(n)}{f(n)}=0. \tag{4}$$

Solution. Let

$$f(n) := n \sum_{k=1}^{n} f_k(n). \tag{5}$$

Then we have for any fixed m, when $n > m$,

$$f(n) = n\,[f_1(n) + \cdots + f_n(n)] > n\, f_m(n). \tag{6}$$

Thus $\lim_{n\to\infty} \frac{f_m(n)}{f(n)} = 0$ as desired.