NAME: ID:

Math 317 Quiz 4 Solutions

MAR. 3, 2014

- The quiz has three problems. Total $10+1$ points. It should be completed in 20 minutes.

QUESTION 1. (5 PTS) *Find a power series* $\sum_{n=1}^{\infty} a_n\,(x-x_0)^n$ *such that* $\forall n\; a_n \neq 0$ *and its radius of convergence* $\neq (\limsup_{n\to\infty} |a_{n+1}/a_n|)^{-1}$.

Solution. $a_n = \begin{cases} 2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$. The radius of convergence is

$$\left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1} = 1 \tag{1}$$

but

$$\left(\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|\right)^{-1} = \frac{1}{2}. \tag{2}$$

QUESTION 2. (5 PTS) *Let* $f(x)$ *be periodic with period* $2\,\pi$ *and equals* x^2 *on* $[-\pi,\pi]$. *Calculate the Fourier expansion of* f *on* $[-\pi,\pi]$.

Solution. As f is even $b_n = 0$. We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,\mathrm{d}x = \frac{2}{3}\,\pi^2. \tag{3}$$

For $n \geqslant 1$ we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(n\,x)\,\mathrm{d}x \\ &= \frac{1}{n\,\pi}\int_{-\pi}^{\pi} x^2\,\mathrm{d}\sin(n\,x) \\ &= -\frac{2}{n\,\pi}\int_{-\pi}^{\pi} \sin(n\,x)\,x\,\mathrm{d}x \\ &= \frac{2}{n^2\,\pi}\int_{-\pi}^{\pi} x\,\mathrm{d}\cos(n\,x) \\ &= \frac{2}{n^2\,\pi}\left[\pi\cos(n\,\pi) - (-\pi)\cos(-n\,\pi) - \int_{-\pi}^{\pi}\cos(n\,x)\,\mathrm{d}x\right] \\ &= \frac{4}{n^2}\,(-1)^n. \end{aligned} \tag{4}$$

Thus the Fourier expansion therefore is

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos(n\,x). \tag{5}$$

QUESTION 3. (1 BONUS PT) *Use the result from the previous problem to prove* $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution. We prove that $f(x)$ is Holder continuous at $x = \pi$. Once this is done the convergence theory of Fourier series yields

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos(n\,\pi) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \tag{6}$$

and the conclusion follows immediately.

Since $f(x)$ has period $2\,\pi$, we have

$$f(x) = \begin{cases} x^2 & x \in [-\pi, \pi] \\ (x - 2\,\pi)^2 & x \in [\pi, 3\,\pi] \end{cases}. \tag{7}$$

Thus for any $x \in (0, 2\,\pi)$ we have

$$\begin{aligned} |f(x) - f(\pi)| &= \begin{cases} |x^2 - \pi^2| & x \in (0, \pi) \\ |(x - 2\,\pi)^2 - \pi^2| & x \in (\pi, 2\,\pi) \end{cases} \\ &= \begin{cases} |x + \pi| \cdot |x - \pi| & x \in (0, \pi) \\ |x - 3\,\pi| \cdot |x - \pi| & x \in (\pi, 2\,\pi) \end{cases} \\ &< 2\,\pi\,|x - \pi|. \end{aligned} \tag{8}$$

Therefore $f(x)$ is Holder continuous at $x = \pi$ and (6) is justified.