Name: ID:

Math 317 Quiz 2 Solutions

Jan. 20, 2014

- The quiz has three problems. Total $10+1$ points. It should be completed in 20 minutes.

Question 1. (5 pts) *Prove the convergence of the series*

$$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3\,n)!}. \tag{1}$$

Proof. We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{((n+1)!)^3/(3\,n+3)!}{(n!)^3/(3\,n)!} = \frac{(n+1)^3}{(3\,n+3)\,(3\,n+2)\,(3\,n+1)}. \tag{2}$$

Thus

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{27} < 1 \tag{3}$$

and convergence follows from ratio test. □

Question 2. (5 pts) *Let $\{a_n\},\{b_n\}$ be two sequences of real numbers satisfying the following:*

i. There is $M>0$ such that for all $n\in\mathbb{N}$, $|\sum_{m=1}^{n} a_m| < M$;

ii. $\lim_{n\to\infty} b_n = 0$.

Does it follow that $\sum_{n=1}^{\infty} a_n\,b_n$ converges? Justify your answer.

Proof.

No. We take $a_n = (-1)^n, b_n = \frac{(-1)^n}{n}$. □

Question 3. (1 bonus pt) *Let $\sum_{n=1}^{\infty} a_n$ be a divergent positive series. Discuss the convergence/divergence of $\sum_{n=1}^{\infty} \frac{a_n}{1+n\,a_n}$. Justify your answer (If it must converge or must diverge, prove; Otherwise give one example of convergence and one of divergence).*

Solution. Taking $a_n = \frac{1}{n}$ we have divergence; Taking $a_n = \begin{cases} \frac{1}{k} & n = k^2 \\ 0 & n \text{ not a square} \end{cases}$ we have divergence of $\sum_{n=1}^{\infty} a_n$ while convergence of $\sum_{n=1}^{\infty} \frac{a_n}{1 + n\, a_n}$ since

$$\frac{a_n}{1 + n\, a_n} = \begin{cases} \dfrac{1}{k\,(k+1)} & n = k^2 \\ 0 & \text{otherwise} \end{cases}. \tag{4}$$

So the series sum up to 1.