NAME: ID:

Math 317 Quiz 1 Solutions

JAN. 8, 2014

- The quiz has three problems. Total $10+1$ points. It should be completed in 20 minutes.

Question 1. (5 pts) *Prove by definition or Cauchy criterion that* $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ *converges and find its value.*

Proof.

- By definition.

  We have

$$S_n = \frac{2}{3} + \cdots + \left(\frac{2}{3}\right)^n = \frac{2}{3}\left[1 + \cdots + \left(\frac{2}{3}\right)^{n-1}\right] = \frac{2}{3} \cdot \frac{1-(2/3)^n}{1-(2/3)} = 2 - 2\cdot\left(\frac{2}{3}\right)^n. \tag{1}$$

  Taking limit $\lim_{n\to\infty} S_n = 2$. Therefore the series converges to 2.

- By Cauchy.

  For any $\varepsilon > 0$, take $N > \log_{3/2}(3/\varepsilon)$, then for any $m > n > N$, we have

$$\begin{aligned}\left|\left(\frac{2}{3}\right)^{n+1} + \cdots + \left(\frac{2}{3}\right)^m\right| &= \left(\frac{2}{3}\right)^{n+1}\left[1 + \cdots + \left(\frac{2}{3}\right)^{m-n-1}\right] \\ &= \left(\frac{2}{3}\right)^{n+1} \cdot \frac{1-(2/3)^{m-n}}{1-(2/3)} \\ &< 3\cdot\left(\frac{2}{3}\right)^{n+1} < 3\cdot\left(\frac{2}{3}\right)^N < \varepsilon. \end{aligned} \tag{2}$$

  Therefore the series is Cauchy and converges. To find its value, we calculate

$$\frac{3}{2}\cdot\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n - \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = 1 \Longrightarrow \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = 2. \tag{3}$$

□

Question 2. (5 pts) *Prove by definition that* $\sum_{n=1}^{\infty} (-1)^n$ *does not converge.*

Proof. We easily calculate:

$$S_n = \begin{cases} -1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}. \tag{4}$$

Assume $\lim_{n \to \infty} S_n = L$. Then there is $N \in \mathbb{N}$ such that for all $n > N$, $|S_n - L| < 1/2$. This implies $|-1 - L| < 1/2$ and $|L| < 1/2$. Contradiction. Therefore $\lim_{n \to \infty} S_n$ does not exist and the original series does not converge. □

Question 3. (1 bonus pt) *Prove that if* a_n *satisfy* $0 < a_n < a_{2n} + a_{2n+1}$ *for all* $n \geqslant 1$, *then* $\sum_{n=1}^{\infty} a_n$ *does not converge to a finite number.*

Proof. We prove that the series is not Cauchy. For any $N \in \mathbb{N}$, take $k \in \mathbb{N}$ such that $2^k > N$. Then we have

$$\begin{aligned} |a_{2^k} + \cdots + a_{2^{k+1}-1}| &= (a_{2^k} + a_{2^k+1}) + (a_{2^k+2} + a_{2^k+3}) + \cdots + (a_{2^{k+1}-2} + a_{2^{k+1}-1}) \\ &> a_{2^{k-1}} + a_{2^{k-1}+1} + \cdots + a_{2^k-1} \\ &> a_{2^{k-2}} + \cdots + a_{2^{k-1}-1} \\ &\vdots \\ &> a_1 > 0. \end{aligned} \tag{5}$$

Thus the series is not Cauchy and cannot converge to a finite number. □