The function $f(x):=\sum_{n=1}^{\infty}\frac{\sin(n\,x)}{n^2}$

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The purpose of this special lecture note is to discuss the continuity, integrability, and differentiability of the function

$$f(x):=\sum_{n=1}^{\infty}\frac{\sin(n\,x)}{n^2}. \tag{1}$$

1. The function is defined for all $x\in\mathbb{R}$.

   Proof. Let $x\in\mathbb{R}$ be arbitrary. Then we have

   $$\left|\frac{\sin(n\,x)}{n^2}\right|\leqslant\frac{1}{n^2}. \tag{2}$$

   As $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, by Comparison Theorem we have the convergence of $\sum_{n=1}^{\infty}\frac{\sin(n\,x)}{n^2}$. □

2. The function is continuous on $\mathbb{R}$.

   Proof. We prove the convergence is uniform on $\mathbb{R}$. This follows immediately from (2), the convergence of $\sum_{n=1}^{\infty}\frac{1}{n^2}$, and Weierstrass' M-test.

   Now since for each fixed n, $\frac{\sin(n\,x)}{n^2}$ is continuous on $\mathbb{R}$, $f(x)$ is also continuous on $\mathbb{R}$. □

3. The function is Riemann integrable on any compact interval $[a,b]\subset\mathbb{R}$, and furthermore

   $$\int_a^b f(x)\,\mathrm{d}x=\sum_{n=1}^{\infty}\int_a^b\frac{\sin(n\,x)}{n^2}\,\mathrm{d}x. \tag{3}$$

   Proof. This follows immediately from the uniform convergence we have just proved. □

4. The function is differentiable at every $x\neq 2\,k\,\pi$ $(k\in\mathbb{Z})$, and furthermore at such x,

   $$f'(x)=\sum_{n=1}^{\infty}\frac{\cos(n\,x)}{n}. \tag{4}$$

   Proof. Since

   $$\left(\frac{\sin(n\,x)}{n^2}\right)'=\frac{\cos(n\,x)}{n}, \tag{5}$$

   all we need to show is the uniform convergence of

   $$\sum_{n=1}^{\infty}\frac{\cos(n\,x)}{n}. \tag{6}$$

   First it is clear that the series does not converge for $x=2\,k\,\pi$ for any $k\in\mathbb{Z}$. Thus in the following we focus on $x\neq 2\,k\,\pi$.

   To prove the convergence we apply Abel's re-summation trick:

   - First we obtain a good formula for

     $$S_n(x):=\cos x+\cdots+\cos(n\,x). \tag{7}$$

We have

$$\begin{aligned} S_n(x) &= \frac{\sin(x/2)}{\sin(x/2)}[\cos x + \cdots + \cos(n\,x)] \\ &= \frac{1}{\sin(x/2)}[\sin(x/2)\cos x + \cdots + \sin(x/2)\cos(n\,x)] \\ &= \frac{1}{2\sin(x/2)}\left[\left(\sin\left(x+\frac{x}{2}\right) - \sin\left(x-\frac{x}{2}\right)\right) + \cdots + \left(\sin\left(n\,x+\frac{x}{2}\right) - \right.\right. \\ &\quad \left.\left.\sin\left(n\,x-\frac{x}{2}\right)\right)\right] \\ &= \frac{\sin(n\,x + x/2)}{2\sin(x/2)} - 1/2. \end{aligned} \tag{8}$$

We see that for any compact interval $[a,b]$ not containing $2\,k\,\pi$, there is $M = M(a,b)$ (that is, depending on a,b – more precisely depending on the distance between a,b and the nearest $2\,k\,\pi$) such that

$$\forall n \in \mathbb{N}, \quad \forall x \in [a,b], \qquad |S_n(x)| < M. \tag{9}$$

- Now we apply the re-summation trick. For any $m > n$, we have

$$\begin{aligned} \left|\frac{\cos((n+1)\,x)}{n+1} + \cdots + \frac{\cos(m\,x)}{m}\right| &= \left|\frac{S_{n+1}(x) - S_n(x)}{n+1} + \cdots + \frac{S_m(x) - S_{m-1}(x)}{m}\right| \\ &= \left|\frac{S_m(x)}{m} - \frac{S_n(x)}{n+1} + S_{n+1}(x)\left(\frac{1}{n+1} - \right.\right. \\ &\quad \left.\left.\frac{1}{n+2}\right) + \cdots + S_{m-1}(x)\left(\frac{1}{m-1} - \frac{1}{m}\right)\right| \\ &\leqslant \frac{M}{m} + \frac{M}{n+1} + M\left[\left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \right. \\ &\quad \left.\left(\frac{1}{m-1} - \frac{1}{m}\right)\right] \\ &= \frac{2\,M}{n+1}. \end{aligned} \tag{10}$$

Note that this holds for every $x \in [a,b]$.

- Finally we prove uniform convergence.

  Taking $m \to \infty$ in the above estimate, we have (denote the limit function by $\phi(x)$)

$$\forall n \in \mathbb{N}, \quad \forall x \in [a,b], \qquad |\phi(x) - S_n(x)| \leqslant \frac{2\,M}{n+1}. \tag{11}$$

Now let $\varepsilon > 0$ be arbitrary. Take $N > \frac{2\,M}{\varepsilon}$. Then for every $n > N$ and every $x \in [a,b]$, we have

$$|\phi(x) - S_n(x)| \leqslant \frac{2\,M}{n+1} < \frac{2\,M}{N} < \varepsilon. \tag{12}$$

Thus $S_n(x) \longrightarrow \phi(x)$ uniformly on $[a,b]$.

Now take any $x \neq 2\,k\,\pi$. There is $a < x < b$ such that $[a,b]$ does not contain any $2\,k\,\pi$. We see that $\sum_{n=1}^{\infty} \frac{\cos(n\,x)}{n}$ converges uniformly on $[a,b]$. Consequently $f(x)$ is differentiable on (a,b) and in particular at x. □

5. The function is not differentiable at every $x = 2\,k\,\pi$ $(k \in \mathbb{Z})$.

Proof. Again thanks to periodicity, all we need to prove is $f'(0)$ does not exist. We achieve this through proving

$$\lim_{m\to\infty}\frac{f(1/m)-f(0)}{1/m}=+\infty. \tag{13}$$

Clearly $f(0)=0$. We have

$$\begin{aligned}\frac{f(1/m)}{1/m} &= \sum_{n=1}^{\infty}\frac{\sin\left(\frac{n}{m}\right)}{n^2/m}\\ &= \sum_{n=1}^{m}\frac{1}{n}\frac{\sin(n/m)}{n/m}+m\sum_{n=m+1}^{\infty}\frac{\sin(n/m)}{n^2}.\end{aligned}$$

We denote the two sums by A and B.

- Estimate of A.

  It is easy to prove that $\frac{\sin x}{x}$ is decreasing on $(0,\pi/2)$. Thus for each term in A we have

  $$\frac{n}{m}\leqslant 1 \Longrightarrow \frac{\sin(n/m)}{n/m}\geqslant\frac{\sin 1}{1}. \tag{14}$$

  Therefore

  $$A\geqslant c\sum_{n=1}^{m}\frac{1}{n} \tag{15}$$

  where $c:=(\sin 1)/1>0$ is a fixed constant.

- Estimate of B.

  We have

  $$|B|\leqslant m\sum_{n=m+1}^{\infty}\frac{1}{n^2}<m\sum_{n=m+1}^{\infty}\frac{1}{(n-1)\,n}=m\sum_{n=m+1}^{\infty}\left[\frac{1}{n-1}-\frac{1}{n}\right]=1. \tag{16}$$

Putting the estimates together, we have

$$\frac{f(1/m)}{1/m}>c\sum_{n=1}^{m}\frac{1}{n}-1 \tag{17}$$

whose limit is obviously ∞ as $m\to\infty$.

Thus we have found a sequence $x_m\longrightarrow 0$ such that

$$\lim_{m\to\infty}\frac{f(x_m)-f(0)}{x_m-0}=+\infty \tag{18}$$

and it follows that f cannot be differentiable at 0. □

Exercise 1. Prove that $\frac{\sin x}{x}$ is decreasing on $(0,\pi)$.

Remark 1. Riemann proposed[1] the following function

$$g(x):=\sum_{n=1}^{\infty}\frac{\sin(n^2 x)}{n^2} \tag{19}$$

1. There is no official record, but Weierstrass stated in a 1875 letter that he "knew" Riemann had constructed this function as early as 1861.

as a candidate for "everywhere continuous but nowhere differentiable" functions. $g(x)$ may look similar to $f(x)$ but the replacement of sin $(n\ x)$ by sin $(n^2\ x)$ totally changed the game. The continuity part is as trivial as that for $f(x)$, but the differentiability part is much more difficult. G. H. Hardy in 1916 prove that $g(x)$ is indeed not differentiable at x when $x/\pi \notin \mathbb{Q}$. Joseph L. Gerver[2] finally proved in 1970/1972 that $g'(x) = -\frac{1}{2}$ at all points of the form $\frac{2\,r+1}{2\,s+1}\,\pi$ where $r, s \in \mathbb{Z}$, and $g(x)$ is not differentiable at every other rational multiple of π. Thus the differentiability of $g(x)$ is completely understood.

The function $g(x)$ played roles in many mathematical fields, including number theory, fractals, and partial differential equations.

Problem 1. Study the continuity of another function proposed by Riemann in his Habilitationsschrift:

$$h(x) := \sum_{n=1}^{\infty} \frac{u(n\,x)}{n^2} \tag{20}$$

where $u(t)$ satisfies:

1. $u(t)$ is periodic with period 1, that is $u(t+1) = u(t)$ for all $t \in \mathbb{R}$.
2. $u\left(\frac{1}{2}\right) = 0$.
3. $u(t) = t$ for $t \in [0, 1/2)$ and $u(t) = t - 1$ for $t \in (1/2, 1)$.

Exercise 2. Plot the partial sums of all the above pathological functions: $f(x), g(x), h(x)$, together with

- The function proposed by Weierstrass:

$$w(x) := \sum_{n=1}^{\infty} a^n \cos(b^n\,\pi\,x) \tag{21}$$

  where $0 < a < 1$, b an odd integer, $a\,b > 1$.[3]

- The function proposed by van der Waerden: See §4.2 of this week (Week 2)'s notes.

for different n's (for example $n = 5, 10, 20, 50, 100$, etc.).

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2. Now at Rutgers University: http://math.camden.rutgers.edu/faculty/.

3. Weierstrass originally required $a\,b > 1 + \frac{3\,\pi}{2}$. Hardy in 1916 relaxed it to $a\,b > 1$. You should try different a, b and see what happens.