Math 317 Week 10: Vector Analysis in Science and Engineering

MARCH 15, 2014

TABLE OF CONTENTS

1. Snake Algorithm and Level Set Method

References.

- Kass, M., Witkin, A, and Terzopoulos, D., *Snakes: Active Contour Models*, Int. J. of Comp. Vision 1, 321 - 331, 1988.
- Osher, Stanley and Fedkiw, Ronald, *Level Set Methods and Dynamic Implicit Surfaces*, Applied Mathematical Sciences 153, Springer 2003. Chapter 12.
- Demo: http://users.ecs.soton.ac.uk/msn/book/new_demo/Snakes/.

1.1. Snake (Active contour)

1.1.1. The effect of minimizing arc length

Theorem 1. *Let $D \subseteq \mathbb{R}^2$ be compact, convex and with C^1 boundary. Let L be a closed curve enclosing D. Then*

$$\text{length}(L) > \text{length}(\partial D). \tag{1}$$

Proof. Since D is convex, if we let $P = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n = \boldsymbol{x}_0\}$ be a partition of ∂D, then connecting $\boldsymbol{x}_0 - \boldsymbol{x}_1 - \cdots - \boldsymbol{x}_n$ gives us a convex polygon D_P inscribed in D. We have seen in previous lectures that

$$\lim_{d(P) \longrightarrow 0} \text{length}(\partial D_P) = \text{length}(\partial D). \tag{2}$$

Together with the fact that $D_P \subseteq D$, we see that it suffices to prove (1) when D is a convex polygon. The proof involves some technicality but is trivial at an informal level. Therefore we simply illustrate the idea as follows,

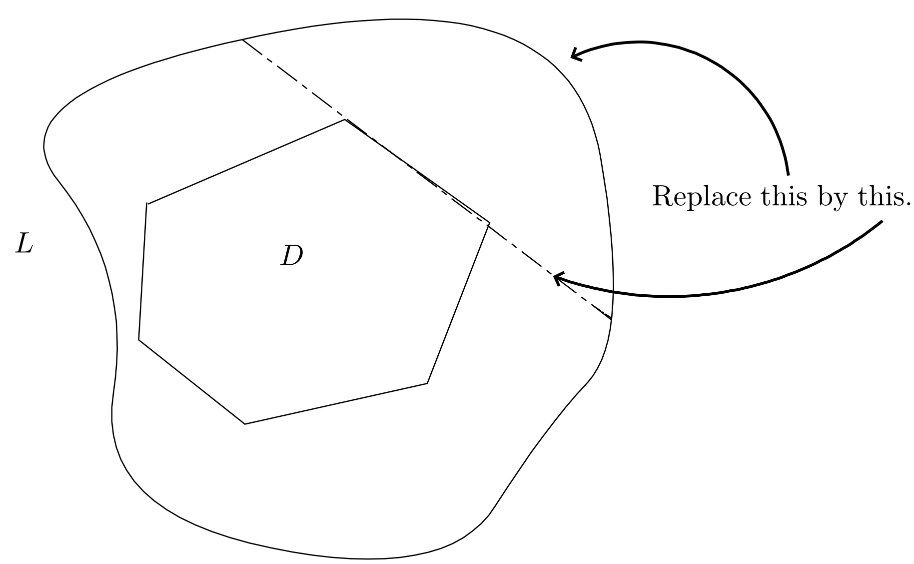

Figure 1. L can be shortened if it is different from ∂D

and leave the details as an exercise. □

Exercise 1. Let $D\subset\mathbb{R}^2$ be compact and convex. Let $\boldsymbol{x}_0\in\partial D$. Prove: There is a unit vector $\boldsymbol{n}$ such that

$$D\subset\{\boldsymbol{x}\in\mathbb{R}^2|\,(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{n}\leqslant 0\}. \tag{3}$$

Then use this to prove Theorem 1 directly. (Hint:[1])

COROLLARY 2. *Let $D\subseteq\mathbb{R}^2$ be compact and with C^1 boundary. Let L be a closed curve enclosing D such that length(L) is the smallest among all such curves. Then*

$$L=\partial(\operatorname{conv}(D)) \tag{4}$$

where

$$\operatorname{conv}(D)=\cap_{E\in W}E, \qquad W:=\{E\subseteq\mathbb{R}^2|\,D\subseteq E, E \text{ convex}\} \tag{5}$$

is the convex hull of D, that is the smallest convex set containing D.

1.1.2. The idea of snake

Images as functions.

An $N\times N$ digital grey-scale image can be identified as a function F from $\{(m,n)|\,m,n=1,2,\ldots,N\}$ to $\{0,1,2,\ldots,255\}$ in the sense that $F(m,n)$ represents the grey-scale value of the image at position (m,n). Now if we "normalize" everything, we can represent the image by a function defined in $[0,1]^2$ and taking values in $[0,1]$:

$$f\Big(\frac{m}{N},\frac{n}{N}\Big):=\frac{1}{256}F(m,n). \tag{6}$$

When N is large enough, we can treat f simply as from $[0,1]^2\mapsto[0,1]$.

Let $u_0(x,y)\colon[0,1]^2\mapsto\mathbb{R}$ be the image. Let $\boldsymbol{x}\colon[0,1]\mapsto\mathbb{R}^2$ be a parametrized curve. We minimize

$$I(\boldsymbol{x}):=\alpha\int_0^1\|\boldsymbol{x}'(s)\|^2\,\mathrm{d}s+\beta\int_0^1\|\boldsymbol{x}''(s)\|^2\,\mathrm{d}s-\lambda\int_0^1\|\nabla u_0(\boldsymbol{x}(s))\|^2\,\mathrm{d}s. \tag{7}$$

where α,β,λ are positive parameters that need to be adjusted for different images. The goal is to find appropriate α, β, λ such that the minimizer is the boundary of regions of interest, such as a human figure.

NOTATION 3. *Here we use "s" instead of "t" because in the study of active snakes, "t" is reserved for time. Note that the s here in general is not the arc length parameter.*

Exercise 2. Explain the purpose of each term in the "energy functional" of the snake. In particular,

1. If there is $\boldsymbol{y}_0\in L\cap D^c$, then there is $\boldsymbol{x}_0\in\partial D$ such that $\|\boldsymbol{x}_0-\boldsymbol{y}_0\|=\operatorname{dist}(\boldsymbol{y}_0,\partial D)>0$. Now take $\boldsymbol{n}=\frac{\boldsymbol{y}_0-\boldsymbol{x}_0}{\|\boldsymbol{y}_0-\boldsymbol{x}_0\|}$ and show that $D\subset\{\boldsymbol{x}\in\mathbb{R}^2|\,(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{n}\leqslant 0\}$.

Functional, Functional derivatives, and Calculus of Variations

A functional is a function of functions. For example, integration over a given interval $[a,b]$ is a functional whose "variables" are all integrable function on $[a,b]$:

$$I(f) := \int_a^b f(x)\,\mathrm{d}x. \tag{8}$$

Just like the derivative of a function at a given point x_0 is a linear function, the "functional derivative" of a functional at a given function $f_0(x)$ is also a linear functional, defined through

$$I(f_0+\delta f) = I(f_0) + [(DI)(f_0)](\delta f) + o(\delta f). \tag{9}$$

(Review 217 lecture notes on differentials if you find (9) hard to understand.) We will not explain the meaning of $o(\delta f)$ here since that can only be done after the introduction of function spaces – vector spaces where each "vector" is a function.

Now for the functional defined in (8), we have

$$I(f_0+\delta f) = \int_a^b f_0(x)\,\mathrm{d}x + \int_a^b \delta f(x)\,\mathrm{d}x \Longrightarrow (DI)(f_0) = \int_a^b \cdot\,\mathrm{d}x = I. \tag{10}$$

Exercise 3. In light of the fact that $I(f)$ is linear, why is $(DI)(f_0) = I$ not surprising?

Functionals like (8) is a bit boring. Things get more interesting when we consider functionals that are nonlinear and involve derivatives. For example

$$U(f) := \frac{1}{2}\int_a^b (f'(x))^2\,\mathrm{d}x \tag{11}$$

where the domain is $\{f \in C^1[a,b], f(a) = f(b) = 0\}$. Now, to make sure $f_0 + \delta f$ still belongs to this set, we need to require $\delta f = 0$ at $x = a, b$.

Exercise 4. Prove that

$$U(f_0+\delta f) = U(f_0) - \int_a^b f_0''(x)\,\delta f(x)\,\mathrm{d}x + \int_a^b (\delta f'(x))^2\,\mathrm{d}x \tag{12}$$

and argue formally that

$$[(DU)(f_0)](v(x)) = -\int_a^b f_0''(x)\,v(x)\,\mathrm{d}x. \tag{13}$$

that is $[(DU)(f_0)] = -\int_a^b f_0''(x) \cdot\,\mathrm{d}x$.

The study of optimization problems for functionals is called "Calculus of Variations". It turns out that, parallel to the theory of optimization of functions, the optimal solution for problems of Calculus of variations is also characterized by the vanishing of the derivative – the functional derivative.

Exercise 5. Argue formally that the solution to $\min U(f)$ over $\{f \in C^1[a,b], f(a) = f(b) = 0\}$ is $f = 0$. Then prove it rigorously.

1.2. Level set formulation and curvature flow

In the level set formulation, curves and surfaces are represented as the zero level set of functions. In the following we focus on the case of curves in the plane. More specifically, when considering the closed curve

$$\boldsymbol{x}(s) : [a,b] \mapsto \mathbb{R}^2, \qquad \boldsymbol{x}(a) = \boldsymbol{x}(b), \tag{14}$$

we find a function $\phi(\boldsymbol{x},t)\colon \mathbb{R}^2\times\mathbb{R}\mapsto\mathbb{R}$ such that

$$\boldsymbol{x}([a,b],t)=\{(\boldsymbol{x},t)|\,\phi(\boldsymbol{x},t)=0\}. \tag{15}$$

Such ϕ is called a "level set function".

Exercise 6. Part of the information in the curves $\boldsymbol{x}(s,t)$ is lost when using the level set representation. Which part? (Hint:[2])

LEMMA 4. *Let $\boldsymbol{x}(t)\colon[a,b]\mapsto\mathbb{R}^N$ be a curve with level set function ϕ. Let $\boldsymbol{v}(\boldsymbol{x},t)\colon\mathbb{R}^N\times\mathbb{R}\mapsto\mathbb{R}^N$ be C^1. Then*

$$\boldsymbol{x}(t)\ \text{satisfies}\ \dot{\boldsymbol{x}}(t)=\boldsymbol{v}(\boldsymbol{x}(t),t)\Longleftrightarrow\frac{\partial\phi}{\partial t}+\boldsymbol{v}\cdot\nabla\phi=0. \tag{16}$$

From this we see that the trace the movement of a curve in the plane, it suffices to solve a partial differential equation.

Why do we do this? Aren't ODEs simpler than PDEs?

It turns out that, numerically solving the ODE system $\dot{\boldsymbol{x}}(s,t)=\boldsymbol{v}(\boldsymbol{x}(s,t),t)$ is usually not a good idea – for example, if we try to trace the trajectory of a collection of points $\{\boldsymbol{x}(s_n,t)\}_{n=1}^N$, then we need to deal with issues such as the crossing of discretized paths. On the other hand, there are many efficient and stable methods to solve the PDE $\frac{\partial\phi}{\partial t}+\boldsymbol{v}\cdot\nabla\phi=0$. More importantly, solving the ODE system preserves the information of the parametrization of the curve at every time t, while such information often is of no value to us.

Exercise 7. Let $\boldsymbol{x}(t)\colon[a,b]\mapsto\mathbb{R}^2$ be C^1 for each t and evolves according to

$$\dot{\boldsymbol{x}}(s,t)=\boldsymbol{v}(\boldsymbol{x}(s,t),t). \tag{17}$$

Here "dot" denotes time derivative, that is $\dot{\boldsymbol{x}}(s,t)=\frac{\partial\boldsymbol{x}}{\partial t}$. Let the velocity field satisfy $\frac{\partial\boldsymbol{x}}{\partial s}(s,t)\parallel\boldsymbol{v}(\boldsymbol{x}(t))$. Prove that the trace of $\boldsymbol{x}(t)$ does not change with time. Think: What changed? (Hint:[3])

Exercise 8. Let $\boldsymbol{x}(t)\colon[a,b]\mapsto\mathbb{R}^2$ be C^1 for each t and evolves according to

$$\dot{\boldsymbol{x}}(s,t)=\boldsymbol{v}(\boldsymbol{x}(s,t),t),\qquad \boldsymbol{x}(s,0)=\boldsymbol{x}_0(s); \tag{18}$$

Let $v_n(\boldsymbol{x},t)=\boldsymbol{v}\cdot\boldsymbol{n}$ be the normal velocity where $\boldsymbol{n}(\boldsymbol{x}(s,t),t)$ is the unit normal vector to the curve at $\boldsymbol{x}(s,t)$. Prove that the traces of

$$\dot{\boldsymbol{y}}(s,t)=v(\boldsymbol{x}(s,t),t)\,\boldsymbol{n}(\boldsymbol{x}(s,t),t),\qquad \boldsymbol{y}(s,0)=\boldsymbol{x}_0(s) \tag{19}$$

coincide with that of $\boldsymbol{x}(s,t)$ at every time t.

The combination the idea of active snake and level set formulation leads to the idea of curvature flow, where the velocity $\boldsymbol{v}=c\,\kappa\,\boldsymbol{n}$ where κ is the curvature of the curve and $\boldsymbol{n}$ the unit normal vector. One realization of this idea is the following equation:[4]

$$\frac{\partial\phi}{\partial t}=g(\nabla u_0)\,\kappa\,|\nabla\phi|+\nabla g(\nabla u_0)\cdot\nabla\phi. \tag{20}$$

Again, although (20) looks formidable, there are efficient numerical methods for this type of equations.

2. The parametrization. In other words, from ϕ we can only recover the trace of the curve. (recall the difference between a curve and its trace!)

3. The parametrization.

4. Zhao, H.-K., Chan, T., Merriman, B., and Osher, S., *A Variational Level Set Approach to Multiphase Motion*, J. Comput. Phys., 127, 179 - 195, 1996.

2. FLYING IN THE WIND

References.

- (VELTKAMP-KLAMKIN) Veltkamp, G. W. and Klamkin, M. S., *Flight in An Irrotational Wind Field (M. S. Klamkin)*, SIAM Rev., 4 (2), 155 - 156.

Setting up the Problem

If an aircraft travels at a constant air speed, and traverses a closed curve in a horizontal place (with respect to the ground), the time taken is always less when there is no wind, than when there is any constant wind. Show that this result is also valid for any irrotational wind field and any closed curve.

Let S be the closed curve that is the trajectory of the aircraft. Let $\boldsymbol{T} = \begin{pmatrix} T_x \\ T_y \end{pmatrix}$ be the unit tangent vector pointing in the direction of the movement of the aircraft. Let $\boldsymbol{N} = \boldsymbol{T}^{\perp} := \begin{pmatrix} -T_y \\ T_x \end{pmatrix}$ be the induced normal vector. Note that this is the opposite of the outer normal vector.

We make the assumption that the engine of the aircraft works at a constant power, that is if there is no wind, the velocity would be $\boldsymbol{v}$ such that $\|\boldsymbol{v}\| = V$ is a constant.

Finally, to simplify presentation, we normalize $V = 1$. In the following we also assume $\|\boldsymbol{w}\| < 1$, that is the wind speed is less than the aircraft speed when there is no wind.

2.1. No wind, constant wind

- No wind.

  When there is no wind, the aircraft obviously directs itself along $\boldsymbol{T}$, that is its velocity is $\boldsymbol{v} = \boldsymbol{T}$. Consequently the time is given by

  $$T(\text{No wind}) = \text{length of } S. \tag{21}$$

- Constant wind

  Now consider the case when the wind is a constant vector $\boldsymbol{w}$. In this case the aircraft must direct itself in a way that

  $$\boldsymbol{v} + \boldsymbol{w} \parallel \boldsymbol{T} \Longleftrightarrow \boldsymbol{v} + \boldsymbol{w} \perp \boldsymbol{N}. \tag{22}$$

  Writing $\boldsymbol{v} = v_T \boldsymbol{T} + v_N \boldsymbol{N}$ and $\boldsymbol{w} = w_T \boldsymbol{T} + w_N \boldsymbol{N}$, we have

  $$v_N = -w_N. \tag{23}$$

  Together with $\|\boldsymbol{v}\| = 1$ we have

  $$v_T = \pm\sqrt{1 - w_N^2}. \tag{24}$$

  To make the travel time as short as possible, obviously $\boldsymbol{v}$ should chosen as

  $$v_T = \sqrt{1 - w_N^2}. \tag{25}$$

Now if we parametrize S by its arc length, we would have

$$T(\boldsymbol{w}) = \int_S (v_T + w_T)^{-1}\, \mathrm{d}s = \int_S \frac{\mathrm{d}s}{w_T + \sqrt{1 - w_N^2}}. \tag{26}$$

Now denote $w = \sqrt{w_T^2 + w_N^2}$ and notice that

$$1 - w_N^2 = (1 - w^2) + w_T^2. \tag{27}$$

This leads to

$$\frac{1}{w_T + \sqrt{1 - w_N^2}} = \frac{\sqrt{(1 - w^2) + w_T^2} - w_T}{1 - w^2}. \tag{28}$$

Exercise 9. Prove that

$$\int_S w_T\, \mathrm{d}s = 0. \tag{29}$$

using the fact that $\boldsymbol{w} = \operatorname{grad} f(x)$ where $f(x) = \boldsymbol{w} \cdot \boldsymbol{x}$. (Hint:[5])

Now we see that, since $1 - w^2 \in [0, 1]$,

$$T(\boldsymbol{w}) = \int_S \frac{\sqrt{(1 - w^2) + w_T^2}}{1 - w^2}\, \mathrm{d}s \geqslant \int_S 1\, \mathrm{d}s = T(\text{No wind}). \tag{30}$$

Exercise 10. Prove that the $\geqslant$ in (30) can be replaced by $>$. (Hint:[6])

2.2. Irrotational wind

Now if we allow non-constant wind, it is obviously possible that the travel time is shorter than the no wind case. For example, we can the wind to be $\boldsymbol{w}(x, y) = w(x, y)\, \boldsymbol{T}(x, y)$ with $w(x, y) \geqslant 0$.

Exercise 11. Prove that for such $\boldsymbol{w}$, there cannot hold

$$\frac{\partial w_1}{\partial y} = \frac{\partial w_2}{\partial x} \qquad \text{for all } (x, y) \in D \tag{31}$$

where w_1, w_2 are the x, y components of $\boldsymbol{w}$, that is $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, and D is the region enclosed by S. (Hint:[7])

Inspired by this, we assume that the wind field is irrotational, that is $\boldsymbol{w}(x, y) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ satisfying

$$\frac{\partial w_1}{\partial y} = \frac{\partial w_2}{\partial x} \qquad \text{for all } (x, y) \in D. \tag{32}$$

We present the two solutions from (VELTKAMP-KLAMKIN) .

5. Parametrize S and then show that the integral equals $f(\boldsymbol{x}(b)) - f(\boldsymbol{x}(a))$, but S is closed which means $\boldsymbol{x}(b) = \boldsymbol{x}(a)$. Or use Green's Theorem.

6. Use Cauchy's MVT to show that there must be a point on S such that $\boldsymbol{T} \parallel \boldsymbol{w}$ there.

7. Green's Theorem.

Solution by G. W. Veltkamp

Let

$$W := w_T + \sqrt{1 - w_N^2}. \tag{33}$$

Then direct calculation verifies

$$\frac{1}{W} = \frac{1}{2}\left(W + \frac{1}{W}\right) - w_T + \frac{w_T^2 + w_N^2}{2\,W} \geqslant 1 - w_T. \tag{34}$$

Now since the wind is irrotational, we have

$$\int_S w_T \,\mathrm{d}s = \int_S \boldsymbol{w} \cdot \mathrm{d}\boldsymbol{l} = \int_D \left[\frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial y}\right] \mathrm{d}(x, y) = 0 \tag{35}$$

and the conclusion immediately follows.

Exercise 12. Verify (34).

Solution by M. S. Klamkin and D. J. Newman (Proposers of this problem)

Notice that

$$T(\boldsymbol{w}) = \int_S \frac{1}{(\boldsymbol{v} + \boldsymbol{w}) \cdot \boldsymbol{T}} \,\mathrm{d}s. \tag{36}$$

Using Cauchy-Schwarz, we have

$$\left[\int_S (\boldsymbol{v} + \boldsymbol{w}) \cdot \boldsymbol{T} \,\mathrm{d}s\right] T(\boldsymbol{w}) \geqslant \left[\int_S 1 \cdot \mathrm{d}s\right]^2 \tag{37}$$

which means

$$T(\boldsymbol{w}) \geqslant \frac{\int_S \mathrm{d}s}{\int_S (\boldsymbol{v} + \boldsymbol{w}) \cdot \boldsymbol{T} \,\mathrm{d}s} T(\text{No wind}). \tag{38}$$

Thus all we need is

$$\int_S (\boldsymbol{v} + \boldsymbol{w}) \cdot \boldsymbol{T} \,\mathrm{d}s \leqslant \int_S \mathrm{d}s. \tag{39}$$

This is obvious.

Exercise 13. Prove (37). (Hint:[8])

Exercise 14. Finish the proof. (Hint:[9])

On the other hand the approach in the previous section, although still works, becomes quite ugly compared to the above two beautiful solutions. In particular, although we still have

$$T(\boldsymbol{w}) = \int_S \frac{\sqrt{(1 - w^2) + w_T^2} - w_T}{1 - w^2} \,\mathrm{d}s \geqslant \int_S 1 \cdot \mathrm{d}s - \int_S \frac{w_T}{1 - w^2} \,\mathrm{d}s. \tag{40}$$

8. $1 = [(\boldsymbol{v} + \boldsymbol{w}) \cdot \boldsymbol{T}]^{1/2} [(\boldsymbol{v} + \boldsymbol{w}) \cdot \boldsymbol{T}]^{-1/2}$.

9. Apply Green's Theorem to the integration of $\boldsymbol{w}$. Then recall that we have assumed $\|\boldsymbol{v}\| = 1$.

Application of Green's Theorem to the second term gives:

$$\begin{aligned}\int_S \frac{w_T}{1-w^2}\,\mathrm{d}s &= \int_S \frac{\boldsymbol{w}}{1-\|\boldsymbol{w}\|^2}\cdot\boldsymbol{T}\,\mathrm{d}s\\ &= \int_S \frac{w_1}{1-w_1^2-w_2^2}\,\mathrm{d}x+\frac{w_2}{1-w_1^2-w_2^2}\,\mathrm{d}y\\ &= \int_D \left[\frac{\partial}{\partial y}\left(\frac{w_2}{1-w_1^2-w_2^2}\right)-\frac{\partial}{\partial x}\left(\frac{w_1}{1-w_1^2-w_2^2}\right)\right]\mathrm{d}(x,y)\\ &= \int_D \frac{2\left(w_2^2\,\frac{\partial w_2}{\partial y}-w_1^2\,\frac{\partial w_1}{\partial x}\right)}{(1-w_1^2-w_2^2)^2}\,\mathrm{d}(x,y)\neq 0.\end{aligned} \tag{41}$$

To make things work we observe that $\int_S w_T\,\mathrm{d}s=0$. Therefore all we need to show is

$$\frac{\sqrt{1-w^2+w_T^2}-w^2\,w_T}{1-w^2}=\frac{\sqrt{1-w^2+w_T^2}-w_T}{1-w^2}+w_T\geqslant 1. \tag{42}$$

This is equivalent to

$$\sqrt{1-w^2+w_T^2}-w^2\,w_T\geqslant 1-w^2\Longleftrightarrow\sqrt{1-w^2+w_T^2}+w^2\,(1-w_T)\geqslant 1. \tag{43}$$

Now denote $b:=\sqrt{1-w^2}$, $x:=w_T$. Then we need to show

$$f(x):=\sqrt{b^2+x^2}-(1-b^2)\,x-b^2\geqslant 0 \tag{44}$$

for all $x\in[-b,b]$ with $b\leqslant 1$.

Exercise 15. Prove that $f''(x)>0$ for all $x\in\mathbb{R}$.

From the formula of $f(x)$ it is clear that we only need to consider $x\in[0,b]$. Taking derivative:

$$f'(x)=\frac{x}{\sqrt{b^2+x^2}}-(1-b^2). \tag{45}$$

Setting $f'(x)=0$ we reach

$$\frac{x}{\sqrt{b^2+x^2}}=(1-b^2)\Longrightarrow x^2=(1-b^2)^2\,(b^2+x^2)\Longleftrightarrow x=\pm\frac{1-b^2}{\sqrt{2-b^2}}. \tag{46}$$

Clearly we should pick $\frac{1-b^2}{\sqrt{2-b^2}}$.

When $(1-b^2)^2\leqslant 1/2$ we have $\frac{1-b^2}{\sqrt{2-b^2}}\leqslant b$ which means $f\left(\frac{1-b^2}{\sqrt{2-b^2}}\right)$ is the minimum. We calculate

$$f\left(\frac{1-b^2}{\sqrt{2-b^2}}\right)=\frac{b^2\left[(2-b^2)-\sqrt{2-b^2}\right]}{\sqrt{2-b^2}}\geqslant 0. \tag{47}$$

Thus we have $f(x)\geqslant 0$ when $(1-b^2)^2\leqslant 1/2$.

When $(1-b^2)^2>1/2$, it suffices to show $f(b)\geqslant 0$, that is

$$\sqrt{2}-1+b^2-b\geqslant 0. \tag{48}$$

But

$$b^2-b+\sqrt{2}-1=(b-1/2)^2+\sqrt{2}-\frac{5}{4}\geqslant\sqrt{2}-\frac{5}{4}>0. \tag{49}$$

Thus we have proved (44) from which (42) follows.

Problem 1. Study the same problem in $\mathbb{R}^3$ or $\mathbb{R}^N$.

3. CONTINUUM MECHANICS

References.

- (TEMAM-MIRANVILLE) Temam, Roger and Miranville, Alain, *Mathematical Modeling in Continuum Mechanics*, Cambridge University Press, 2001. Chapters 1 – 6.

3.1. Kinematics

3.1.1. Basic concepts and relations

What is Kinematics

Kinematics studies motion of objects without considering the cause of motion. For example, consider the motion of a rigid rod with ends $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$. Then its kinematics is governed by the fact that $\|\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)\| = \|\boldsymbol{x}_1(0) - \boldsymbol{x}_2(0)\|$ for all t. This holds no matter what the cause of the motion is.

Exercise 16. Prove that

$$[\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)] \cdot [\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)] = 0 \tag{50}$$

for all t. (Hint:[10])

We consider a collection of material bodies occupying a domain (a connect open set) of the space $\mathbb{R}^3$. Then if it is moving, the domain will change with time. Thus we denote it by Ω_t. In many situations, there is a natural "starting time" t_0, which can often be set as 0. If we take the coordinate fram at $t = 0$ as our reference frame, then the movement of all the material bodies can be represented by a "deformation map":

$$\Phi: \Omega_0 \mapsto \mathbb{R}^3 \tag{51}$$

with the position at time t of the material body taking position $\boldsymbol{a}$ at $t = 0$ given by $\Phi(\boldsymbol{a}, t)$. We make the following assumptions:

- Φ is one-to-one;
- Φ is as smooth as we need;
- $\det \nabla\Phi > 0$ for every $\boldsymbol{a} \in \Omega_0$ and every $t > 0$.

Exercise 17. Find a $\boldsymbol{f}: \mathbb{R}^2 \mapsto \mathbb{R}^2$ such that $\det\nabla\boldsymbol{f} > 0$ everywhere but $\boldsymbol{f}$ is not one-to-one. (Hint:[11])

DEFINITION 5. (DISPLACEMENT) *The function $\boldsymbol{u}: \Omega_0 \mapsto \mathbb{R}^3$ defined by*

$$\boldsymbol{u}(\boldsymbol{a}, t) = \Phi(\boldsymbol{a}, t) - \boldsymbol{a} \tag{52}$$

is called the "displacement map.".

DEFINITION 6. (TRAJECTORY) *For each fixed $\boldsymbol{a} \in \Omega_0$, the curve*

$$t \mapsto \Phi(\boldsymbol{a}, t) \tag{53}$$

is called the "trajectory" of the particle starting at $\boldsymbol{a}$.

10. Differentiate $\|\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)\|^2$.
11. Identify $\mathbb{R}^2$ with $\mathbb{C}$. Consider $z \mapsto z^n$ for $n = 2$.

Now if we following this particular particle, then clearly its velocity at time t is given by $\frac{\partial \Phi}{\partial t}(\boldsymbol{a}, t)$. However, in the context of continuum mechanics, due to the large number of particles we would like to avoid tracing every particle. Therefore we would like to write the velocity using the coordinate system at time t, that is we write $\boldsymbol{v}(\boldsymbol{x}, t)$ instead of $\boldsymbol{v}(\boldsymbol{a}, t)$. Thus we have the relation

$$\boldsymbol{v}(\Phi(\boldsymbol{a}, t), t) = \frac{\partial \Phi(\boldsymbol{a}, t)}{\partial t}. \tag{54}$$

Now we calculate acceleration:

$$\frac{\partial^2 \Phi(\boldsymbol{a}, t)}{\partial t^2} = \frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{v}(\Phi(\boldsymbol{a}, t), t)] = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}. \tag{55}$$

Exercise 18. Where are the functions on the RHS evaluated at? (Hint:[12])

3.1.2. Eulerian and Lagrangian

In (55) we have seen that, for the same quantity "acceleration at time t of the particle originally at $\boldsymbol{a}$" can be represented in two ways: $\frac{\partial^2 \Phi(\boldsymbol{a}, t)}{\partial t^2}$, $\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$. The difference between these two formulas is that, the former uses the information of the particle trajectory while the latter does not.

DEFINITION 7. *A function associated with a particle M can be written as $g(\boldsymbol{a}, t)$ or $h(\boldsymbol{x}, t)$. We call the former Lagrangian representation and the latter Eulerian representation of the same function.*

DEFINITION 8. (MATERIAL DERIVATIVE) *The "material derivative" D_t is defined as*

$$D_t f := \frac{\partial f}{\partial t} + (\boldsymbol{v} \cdot \nabla) f. \tag{56}$$

Thus we can write the accerlation as $D_t \boldsymbol{v}$. Note that D_t can be calculated without tracing particles.

Exercise 19. Let $h(\boldsymbol{x}, t)$ and $g(\boldsymbol{a}, t)$ be the Eulerian and Lagrangian representations of the same quantity related to a certain property of the particles. Prove that

$$\frac{\partial g}{\partial t}(\boldsymbol{a}, t) = \left[\frac{\partial h}{\partial t} + (\boldsymbol{v} \cdot \nabla) h\right](\boldsymbol{x}, t). \tag{57}$$

where $\boldsymbol{x} = \Phi(\boldsymbol{a}, t)$.

3.1.3. Study aggregates of particles

We consider the movement of particles governed by $\Phi(\boldsymbol{a}, t)\colon \Omega_0 \mapsto \Omega_t$, with induced velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$. Let some property of the particles be represented as

$$K(t) := \int_{\Omega_t} C(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}. \tag{58}$$

We would like to derive a differential equation for $K(t)$.

LEMMA 9. *Let $A(t)\colon \mathbb{R} \mapsto \mathbb{R}^{N \times N}$ be nonsingular for every t. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det A(t)) = [\det A(t)] \cdot \mathrm{tr}\left[\frac{\mathrm{d}A(t)}{\mathrm{d}t} \cdot A(t)^{-1}\right]. \tag{59}$$

Proof. Wlog we calculate the derivative at $t = 0$. Denote $A_0 = A(0)$. Then we have

$$\det A(t) - \det A(0) = (\det A_0)\left[\det\left(A(t)\, A_0^{-1}\right) - 1\right]. \tag{60}$$

12. $(\Phi(\boldsymbol{a}, t), t)$.

Thus it suffices to prove the special case $A_0 = I$, which reduces to

$$\lim_{t\to 0} \frac{\det(I + B(t)) - 1}{t} = \operatorname{tr}(B'(0)) = \lim_{t\longrightarrow 0} \operatorname{tr}\left(\frac{B(t)}{t}\right) \tag{61}$$

where $B(0) = 0$. Recalling the calculation formula for determinant:

$$\det C = \sum_{\sigma} (-1)^{\sigma} c_{\sigma(1)\cdots\sigma(N)} \tag{62}$$

where the sum goes over all permutations of $\{1, 2, \ldots, N\}$, we see that

$$\det(I + B(t)) = 1\cdots 1 + \sum_{i=1}^{N} b_{ii}(t)\, 1\cdots 1 + O() \tag{63}$$

□

PROPOSITION 10. *We have*

$$K'(t) = \int_{\Omega_t} \frac{\partial C}{\partial t}\, \mathrm{d}\boldsymbol{x} + \int_{\Omega_t} \operatorname{div}(C\boldsymbol{v})\, \mathrm{d}\boldsymbol{x} = \int_{\Omega_t} \frac{\partial C}{\partial t}\, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_t} C\,(\boldsymbol{v}\cdot\boldsymbol{n})\, \mathrm{d}S. \tag{64}$$

Proof. Exercise. (Hint:[13]) □

Conservation of mass

By Proposition 10 we have the following relation representing mass conservation:

$$0 = M'(t) = \int_{\Omega_t} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\boldsymbol{v})\, \mathrm{d}x. \tag{65}$$

Since this holds for every possible Ω_t, we obtain the equation for mass conservation:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\boldsymbol{v}) = 0. \tag{66}$$

In many situations $\rho(x, t) = \rho_0$ is a constant. Then the conservation of mass reduces to the incompressibility condition $\operatorname{div}\boldsymbol{v} = 0$.

Exercise 20. If

$$K(t) = \int_{\Omega_t} C(\boldsymbol{x}, t)\, \rho(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}, \tag{67}$$

then

$$K'(t) = \int_{\Omega} (D_t C)\, \rho\, \mathrm{d}\boldsymbol{x} \tag{68}$$

where $D_t = \partial_t + \boldsymbol{v}\cdot\nabla$ is the material derivative. (Hint:[14])

Exercise 21. Apply Proposition 10 and Exercise 20 to the evolution of energy, momentum, and angular momentum:

$$\frac{1}{2}\int_{\Omega_t} \rho\, \|\boldsymbol{v}\|^2\, \mathrm{d}\boldsymbol{x}, \qquad \frac{1}{2}\int_{\Omega_t} \rho\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}, \qquad \int_{\Omega_t} \rho\, (\boldsymbol{x}\times\boldsymbol{v})\, \mathrm{d}\boldsymbol{x}. \tag{69}$$

3.2. Dynamics

What is Dynamics

In contrast to Kinematics, dynamics takes into account the causes of motion – the forces.

13. Apply change of variables to reduce the integral to an integral on Ω_0. Note that we have assumed $\det(\nabla_a \Phi) > 0$.
14. Use Proposition 10 then use conservation of mass.

3.2.1. Internal force and stress tensor

We consider two small material bodies sharing part of the boundaries (which is a piece of surface). The assumption now is that the force caused by their interaction depends only on the location $\boldsymbol{x}$ and the normal $\boldsymbol{n}$ of the separating surface. We denote it by $T(\boldsymbol{x}, \boldsymbol{n})\colon \Omega_t \times S^2 \mapsto \mathbb{R}^3$.

Now we study the properties of T. It turns out that T enjoys many symmetries. First, from Newton's 3rd law, we have

$$T(\boldsymbol{x}, \boldsymbol{n}) = -T(\boldsymbol{x}, -\boldsymbol{n}). \tag{70}$$

Furthermore,

Proposition 11. *If T is continuous as a function of $\boldsymbol{x}$ for each fixed $\boldsymbol{n}$, then T is linear in $\boldsymbol{n}$. Thus there is a matrix function $\Sigma(\boldsymbol{x})$ such that*

$$T(\boldsymbol{x}, \boldsymbol{n}) = \Sigma(\boldsymbol{x}) \cdot \boldsymbol{n}. \tag{71}$$

Σ is often called the "stress tensor".

Proof. Take any $\boldsymbol{n} \in S^2$. Wlog let $n_x, n_y, n_z > 0$. We consider the tetrahedron obtained from cutting $\{x, y, z > 0\}$ by the plane $\{\boldsymbol{x} \in \mathbb{R}^3 | \, \boldsymbol{n} \cdot (\boldsymbol{x} - h\, \boldsymbol{n}) = 0\}$ which has normal $\boldsymbol{n}$ and is distance h away from the origin.

Then since its acceleration cannot be infinity, the boundary force must sum up to $O(h)$. Simple calculation gives

$$T(\boldsymbol{x}, \boldsymbol{e}_1)\, n_x + T(\boldsymbol{x}, \boldsymbol{e}_2)\, n_y + T(\boldsymbol{x}, \boldsymbol{e}_3)\, n_z = T(\boldsymbol{x}, \boldsymbol{n}) \tag{72}$$

and the conclusion follows. $\square$

Exercise 22. Prove that Σ is symmetric. (Hint:[15])

3.2.2. Equations of dynamics

Now following basic laws of Newtonian mechanics, we have the equation for conservation of momentum

$$\rho\, D_t \boldsymbol{v} = \operatorname{div}(\Sigma) + \boldsymbol{f}. \tag{74}$$

Here $\boldsymbol{f}$ represents external force, while the divergence of the tensor Σ is defined as

$$\operatorname{div}(\Sigma) = \partial_j \Sigma_{ij}. \tag{75}$$

Note that Einstein's summation notation is used here.

Pressure

If we make the assumption that there is no "shear" force, that is $T(\boldsymbol{x}, \boldsymbol{n}) \parallel \boldsymbol{n}$ for all $\boldsymbol{n}$, then it can be proved that there is a function $p(\boldsymbol{x})$ such that

$$\Sigma(\boldsymbol{x}) = p(\boldsymbol{x})\, I \tag{76}$$

whre I is the identity matrix.

Exercise 23. Prove that

- $\Sigma\, \boldsymbol{n} \parallel \boldsymbol{n} \Longrightarrow \Sigma = p\, I$; (Hint:[16])
- $\operatorname{div}(p(\boldsymbol{x})\, I) = \nabla p(\boldsymbol{x})$.

---

15. Consider angular momentum: By Gauss's Theorem,

$$0 = \int_{\partial\Omega} \boldsymbol{x} \times (\Sigma(\boldsymbol{x}) \cdot \boldsymbol{n})\, \mathrm{d}S = \int_{\Omega} \epsilon_{ijk} \left[\Sigma_{kj} + x_j\, \Sigma_{kl,l}\right] \mathrm{d}\boldsymbol{x}. \tag{73}$$

Now "shrink" Ω to origin.

16. Assume the contrary: $\Sigma\, \boldsymbol{n}_1 = a_1\, \boldsymbol{n}_1$, $\Sigma\, \boldsymbol{n}_2 = a_2\, \boldsymbol{n}_2$ with $a_1 \neq a_2$. Consider linear combinations of $\boldsymbol{n}_1, \boldsymbol{n}_2$.

3.2.3. Stress-strain (constitutive) relations

The equation of dynamics (together with conservation of mass) cannot really solved because there are many more unknowns than equations. To make them solvable, we need to introduce the so-called stress-strain relations, relating the motion of particles with the stress tensor.

Solids

A material is called "solid" if the stress is produced by the deformation of the material. To quantify this we introduce the right Cauchy-Green tensor

$$C := (\nabla_a \Phi)^T (\nabla_a \Phi) \tag{77}$$

and the deformation tensor (Cauchy-Lagrange tensor)

$$X := \frac{1}{2}(C - I). \tag{78}$$

Thus the material is solid if we can safely assume

$$\Sigma(\boldsymbol{a}, t) = f(X). \tag{79}$$

If we denote $U(\boldsymbol{a}, t) = \Phi(\boldsymbol{a}, t) - \boldsymbol{a}$, then when the deformation is small, we have

$$X \sim \frac{1}{2}(\nabla_a U + \nabla_a U^T). \tag{80}$$

Thus $\Sigma(\boldsymbol{a}, t) = f(\nabla_a U + \nabla_a U^T)$. Note that it is wrong to use $f(\nabla_a U)$ as this formula does not have the appropriate symmetry, such as frame-independence.

Exercise 24. Let $\boldsymbol{a}(s): s \in [c, d]$ be a curve in Ω_0. It is mapped to another curve $\Phi(\boldsymbol{a}(s), t)$ in Ω_t. Prove that

$$\mathrm{length}(\Phi(\boldsymbol{a}(s))) = \int_c^d \sqrt{\boldsymbol{a}'(s) \cdot [C(\boldsymbol{a}(s), t)\, \boldsymbol{a}'(s)]}\, \mathrm{d}s. \tag{81}$$

Thus C characterizes the stretching of the line elements.

Exercise 25. Prove

$$\frac{\partial X}{\partial t} = \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T). \tag{82}$$

Note that the RHS is Eulerian.

Fluids

A material is called "fluid" if the stress is produced by the rate of deformation. Therefore for fluids we model

$$\Sigma(\boldsymbol{x}, t) = f(D) \tag{83}$$

where $D = \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)$.

A fluid is called "Newtonian" if f is linear. In this case it can be shown (using thermo-dynamics) that

$$\Sigma = \mu D + \lambda (\operatorname{div} \boldsymbol{v}) I - p I \tag{84}$$

which leads to the Navier-Stokes equations:

$$\rho D_t \boldsymbol{v} = -\nabla p + \mu \triangle \boldsymbol{u} + (\lambda + \mu) \nabla(\operatorname{div} \boldsymbol{v}). \tag{85}$$

Remark 12. From the above we see that it is natural to study solids in a Lagrangian formulation while fluids in a Eulerian formulation.

4. Advanced Topics, Notes, and Comments

4.1. Differentiation inside integration

In many situations we need to calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D f(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} \tag{86}$$

where $D \subseteq \mathbb{R}^N$ is bounded or unbounded. In the framework of Riemann integration we have restrict ourselves to the case where D is bounded.

THEOREM 13. *Let $D \subseteq \mathbb{R}^N$ be a compact set. Let $f(\boldsymbol{x},t)\colon D \times (a,b)$ be C^1. Denote $\varphi(\boldsymbol{x},t) := \frac{\partial f(\boldsymbol{x},t)}{\partial t}$. Let $t_0 \in (a,b)$. Then*

$$F(t) := \int_D f(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} \tag{87}$$

is differentiable at $t = t_0$ and furthermore

$$F'(t_0) = \int_D \varphi(\boldsymbol{x},t_0)\,\mathrm{d}\boldsymbol{x}. \tag{88}$$

Proof. We calculate

$$\frac{F(t)-F(t_0)}{t-t_0} - \int_D \varphi(\boldsymbol{x},t_0)\,\mathrm{d}\boldsymbol{x} = \int_D \left[\frac{f(\boldsymbol{x},t)-f(\boldsymbol{x},t_0)}{t-t_0} - \varphi(\boldsymbol{x},t_0)\right]\mathrm{d}\boldsymbol{x}. \tag{89}$$

Now since $\varphi(\boldsymbol{x},t)$ is continuous on $D \times (a,b)$, there is $\delta > 0$ such that $\varphi(\boldsymbol{x},t)$ is continuous on $D \times [t_0 - \delta_0, t_0 + \delta_0]$. Since D is compact, by Heine-Borel it is closed and bounded. Therefore $D \times [t_0 - \delta_0, t_0 + \delta_0]$ is also compact. Consequently $\varphi(\boldsymbol{x},t)$ is uniformly continuous on $D \times [t_0 - \delta, t_0 + \delta]$.

Now for any $\varepsilon > 0$, take $\delta > 0$ such that

$$\forall (\boldsymbol{x}_1, t_1), (\boldsymbol{x}_2, t_2) \in D \times [t_0 - \delta_0, t_0 + \delta_0], \qquad \|(\boldsymbol{x}_1, t_1) - (\boldsymbol{x}_2, t_2)\| < \delta \Longrightarrow |\varphi(\boldsymbol{x}_1, t_1) - \varphi(\boldsymbol{x}_2, t_2)| < \varepsilon. \tag{90}$$

We have by MVT, for any $|t - t_0| < \delta$,

$$\forall \boldsymbol{x} \in D, \qquad \left|\frac{f(\boldsymbol{x},t)-f(\boldsymbol{x},t_0)}{t-t_0} - \varphi(\boldsymbol{x},t_0)\right| < \varepsilon. \tag{91}$$

Thus

$$\frac{f(\boldsymbol{x},t)-f(\boldsymbol{x},t_0)}{t-t_0} - \varphi(\boldsymbol{x},t_0) \longrightarrow 0 \tag{92}$$

uniformly and consequently

$$\frac{F(t)-F(t_0)}{t-t_0} - \int_D \varphi(\boldsymbol{x},t_0)\,\mathrm{d}\boldsymbol{x} \longrightarrow 0. \tag{93}$$

Thus ends the proof. $\square$

4.2. Electric field of a distribution of electric charges

Let $D \subset \mathbb{R}^3$ be compact and ∂D consists of finitely many pieces of C^1 surfaces. Consider a continuous distribution of electric charges: $\rho\colon D \mapsto \mathbb{R}^+ \cup \{0\}$. Then the electric potential $u(\boldsymbol{x})$ is given by

$$u(\boldsymbol{x}) = \int_D \frac{\rho(\boldsymbol{y})}{\|\boldsymbol{x}-\boldsymbol{y}\|}\,\mathrm{d}\boldsymbol{y}. \tag{94}$$

Note that $u(\boldsymbol{x})$ is defined on the whole $\mathbb{R}^3$. Recall that the electric field $\boldsymbol{E}(\boldsymbol{x})$ is defined as

$$\boldsymbol{E}(\boldsymbol{x}) := -\operatorname{grad} u(\boldsymbol{x}). \tag{95}$$

THEOREM 14. *Assume that ρ is C^2 and $\operatorname{supp}\rho \subset D^o$, where the "support" $\operatorname{supp}\rho := \overline{\{\boldsymbol{x} \mid \rho(\boldsymbol{x}) > 0\}}$. Then we have*

$$\operatorname{div}\boldsymbol{E} = 4\,\pi\,\rho \tag{96}$$

satisfied at all $\boldsymbol{x} \in \mathbb{R}^3$.

Proof. For simplicity of presentation we prove (96) at $\boldsymbol{x} = 0$. Since $\operatorname{supp}\rho$ is compact, dist($\operatorname{supp}\rho$, $\partial D) > 0$. Therefore for $\|\boldsymbol{x}\| < \delta$ we can write

$$u(\boldsymbol{x}) = \int_D \frac{\rho(\boldsymbol{x}+\boldsymbol{z})}{\|\boldsymbol{z}\|}\,\mathrm{d}\boldsymbol{z}. \tag{97}$$

Now as ρ is assumed to be C^2, $u(\boldsymbol{x})$ as defined through (97) is also C^2 due to Theorem 13. Therefore we have

$$\operatorname{div}\boldsymbol{E} = \operatorname{div}(-\operatorname{grad} u) = \int_D \frac{\operatorname{div}_x(-\operatorname{grad}_x \rho(\boldsymbol{x}+\boldsymbol{z}))}{\|\boldsymbol{z}\|}\,\mathrm{d}\boldsymbol{z}. \tag{98}$$

Here $\operatorname{div}_x, \operatorname{grad}_x$ indicate that the differentiations are with respect to $\boldsymbol{x}$. By chain rule we see that

$$\operatorname{div}_x(-\operatorname{grad}_x \rho(\boldsymbol{x}+\boldsymbol{z})) = \operatorname{div}_z(-\operatorname{grad}_z \rho(\boldsymbol{x}+\boldsymbol{z})). \tag{99}$$

Thus we have

$$\operatorname{div}\boldsymbol{E}|_{\boldsymbol{x}=0} = \int_D \frac{\operatorname{div}(-\operatorname{grad}\rho(\boldsymbol{z}))}{\|\boldsymbol{z}\|}\,\mathrm{d}\boldsymbol{z}. \tag{100}$$

Now denote

$$D_\varepsilon := D \cap \{\|\boldsymbol{x}\| > \varepsilon\}. \tag{101}$$

Invoking spherical coordinates, we can prove that

$$\int_D \frac{\operatorname{div}(-\operatorname{grad}\rho(\boldsymbol{z}))}{\|\boldsymbol{z}\|}\,\mathrm{d}\boldsymbol{z} = \lim_{\varepsilon\to 0+}\int_{D_\varepsilon} \frac{\operatorname{div}(-\operatorname{grad}\rho(\boldsymbol{z}))}{\|\boldsymbol{z}\|}\,\mathrm{d}\boldsymbol{z}. \tag{102}$$

Now apply Gauss's Theorem to obtain (note that $\operatorname{grad}\rho = 0$ along ∂D)

$$\begin{aligned}
\int_{D_\varepsilon} \frac{\operatorname{div}(-\operatorname{grad}\rho(\boldsymbol{z}))}{\|\boldsymbol{z}\|}\,\mathrm{d}\boldsymbol{z} &= \int_{D_\varepsilon}\left[\operatorname{div}\left(\frac{-\operatorname{grad}\rho}{\|\boldsymbol{z}\|}\right) + \operatorname{grad}\rho\cdot\operatorname{grad}\left(\frac{1}{\|\boldsymbol{z}\|}\right)\right]\mathrm{d}z \\
&= \int_{\partial D_\varepsilon} \frac{-\operatorname{grad}\rho(\boldsymbol{z})}{\|\boldsymbol{z}\|}\cdot\left(-\frac{\boldsymbol{z}}{\|\boldsymbol{z}\|}\right)\mathrm{d}S + \int_{D_\varepsilon}\operatorname{div}\left(\rho\operatorname{grad}\left(\frac{1}{\|\boldsymbol{z}\|}\right)\right)\mathrm{d}z \\
&\quad -\int_{D_\varepsilon}\rho\operatorname{div}\left(\operatorname{grad}\frac{1}{\|\boldsymbol{z}\|}\right)\mathrm{d}\boldsymbol{z} \\
&= \int_{\partial D_\varepsilon}\frac{\operatorname{grad}\rho(\boldsymbol{z})}{\|\boldsymbol{z}\|}\cdot\frac{\boldsymbol{z}}{\|\boldsymbol{z}\|}\,\mathrm{d}S + \int_{\partial D_\varepsilon}\rho(\boldsymbol{z})\operatorname{grad}\left(\frac{1}{\|\boldsymbol{z}\|}\right)\cdot\boldsymbol{n}\,\mathrm{d}S \\
&= \varepsilon^{-2}\int_{\|\boldsymbol{z}\|=\varepsilon}(\operatorname{grad}\rho(\boldsymbol{z}))\cdot\boldsymbol{z}\,\mathrm{d}S + \int_{\|\boldsymbol{z}\|=\varepsilon}\rho(\boldsymbol{z})\frac{1}{\|\boldsymbol{z}\|^2}\,\mathrm{d}S \\
&= \varepsilon^{-2}\int_{\|\boldsymbol{z}\|=\varepsilon}(\operatorname{grad}\rho(\boldsymbol{z}))\cdot\boldsymbol{z}\,\mathrm{d}S + \varepsilon^{-2}\int_{\|\boldsymbol{z}\|=\varepsilon}\rho(\boldsymbol{z})\,\mathrm{d}S.
\end{aligned} \tag{103}$$

Taking $\varepsilon \longrightarrow 0+$ we have

$$\int_D \frac{\operatorname{div}(-\operatorname{grad}\rho(\boldsymbol{z}))}{\|\boldsymbol{z}\|}\,\mathrm{d}\boldsymbol{z} = 4\,\pi\,\rho(0). \tag{104}$$

Thus ends the proof. ☐

Exercise 26. Prove (102).

Exercise 27. Prove $\operatorname{div}\left(\operatorname{grad}\frac{1}{\|\boldsymbol{z}\|}\right)=0$ for every $\boldsymbol{z}\neq 0$.

Exercise 28. Prove that

$$\lim_{\varepsilon\longrightarrow 0+}\varepsilon^{-2}\int_{\|\boldsymbol{z}\|=\varepsilon}(\operatorname{grad}\rho(\boldsymbol{z}))\cdot\boldsymbol{z}\,\mathrm{d}S=0;\qquad \lim_{\varepsilon\longrightarrow 0+}\varepsilon^{-2}\int_{\|\boldsymbol{z}\|=\varepsilon}\rho(\boldsymbol{z})\,\mathrm{d}S=4\,\pi\,\rho(0). \tag{105}$$

Exercise 29. Prove (96) at general $\boldsymbol{x}$.

Exercise 30. Does the equation still hold if we drop the boundedness assumption on D?

Remark 15. (96) still holds even when ρ is not smooth. The proof is possible in the framework of Riemann integration but much easier in the framework of Lebesgue integration.

Remark 16. From this we can quickly obtain two of the four equations in the system of Maxwell's equations:

$$\operatorname{div}\boldsymbol{E}=4\,\pi\,\rho;\qquad \operatorname{div}\boldsymbol{B}=0. \tag{106}$$

Exercise 31. Assume the Maxwell's equations:

$$\operatorname{div}\boldsymbol{E}=4\,\pi\,\rho;\qquad \operatorname{div}\boldsymbol{B}=0;\qquad \operatorname{curl}\boldsymbol{E}=-\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t};\qquad \operatorname{curl}\boldsymbol{B}=\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}+\frac{4\,\pi}{c}\boldsymbol{J}. \tag{107}$$

Prove the following.

$$\frac{\partial\rho}{\partial t}=-\operatorname{div}\boldsymbol{J}; \tag{108}$$

$$\frac{\partial^2\boldsymbol{E}}{\partial t^2}=c^2\,\triangle\boldsymbol{E};\qquad \frac{\partial^2\boldsymbol{B}}{\partial t^2}=c^2\,\triangle\boldsymbol{B}. \tag{109}$$

4.3. Some technical topics in continuum mechanics

4.3.1. Rigid motion

Let $\boldsymbol{v}:\mathbb{R}^3\mapsto\mathbb{R}^3$ be a velocity field that induces a rigid motion, that is if for any $\Omega_0\subseteq\mathbb{R}^3$,

$$\frac{\partial\Phi(\boldsymbol{a},t)}{\partial t}=\boldsymbol{v}(\Phi(\boldsymbol{a},t),t),\qquad\qquad \boldsymbol{a}\in\Omega_0, t>0 \tag{110}$$

with $\Phi(\boldsymbol{a},0)=\boldsymbol{a}$ for all $\boldsymbol{a}\in\Omega_0$, then for any $\boldsymbol{a}_1,\boldsymbol{a}_2\in\Omega_0$,

$$\|\boldsymbol{a}_1-\boldsymbol{a}_2\|=\|\Phi(\boldsymbol{a}_1,t)-\Phi(\boldsymbol{a}_2,t)\| \tag{111}$$

for all $t>0$.

Exercise 32. Prove that $\boldsymbol{v}$ induce a rigid motion if and only if

$$\forall\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3,\qquad (\boldsymbol{x}-\boldsymbol{y})\cdot[\boldsymbol{v}(\boldsymbol{x},t)-\boldsymbol{v}(\boldsymbol{y},t)]=0 \tag{112}$$

PROPOSITION 17. (STRUCTURE OF RIGID MOTION) *$\boldsymbol{v}$ induced a rigid motion if and only if there are functions $\omega_1(t),\omega_2(t),\omega_3(t)$ and $b(t)$ such that*

$$\boldsymbol{v}(\boldsymbol{x},t)=B(t)\,\boldsymbol{x}+b(t) \tag{113}$$

Note that $\boldsymbol{v}$ is linear in $\boldsymbol{x}$!

where $B(t):=\begin{pmatrix}0&-\omega_3&\omega_2\\ \omega_3&0&-\omega_1\\ -\omega_2&\omega_1&0\end{pmatrix}$.

Exercise 33. Prove that $B(t)\,\boldsymbol{x}=\boldsymbol{\omega}\times\boldsymbol{x}$ where $\boldsymbol{\omega}:=\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}$.

Proof. That any $\boldsymbol{v}$ given by (113) induces a rigid motion is trivial to prove and left as exercise.

Also notice that we could safely ignor the t dependence. Thus all we need to prove is that

$$\forall \boldsymbol{x}, \boldsymbol{y}, \quad (\boldsymbol{x}-\boldsymbol{y}) \cdot [\boldsymbol{v}(\boldsymbol{x})-\boldsymbol{v}(\boldsymbol{y})] = 0 \Longrightarrow \boldsymbol{v}(\boldsymbol{x},t) = B\,\boldsymbol{x} + b. \tag{114}$$

with B taking the above particular form. We further simplify by taking $\boldsymbol{u}(\boldsymbol{x}) := \boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{v}(\boldsymbol{0})$. Then we have

$$\forall \boldsymbol{x}, \quad \boldsymbol{x}\cdot\boldsymbol{u}(\boldsymbol{x}) = 0 \qquad \text{and} \qquad \forall \boldsymbol{x}, \boldsymbol{y}, \qquad (\boldsymbol{x}-\boldsymbol{y})\cdot[\boldsymbol{u}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{y})] = 0 \tag{115}$$

Now taking $\boldsymbol{y} = \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ where $\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, we have

$$\boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{e}_i = -\boldsymbol{u}(\boldsymbol{e}_i)\cdot\boldsymbol{x}. \tag{116}$$

This clearly shows $\boldsymbol{u}(\boldsymbol{x})$ is linear and thus equals $B\,\boldsymbol{x}$ for some matrix B.

Now from $\boldsymbol{x}\cdot(B\,\boldsymbol{x}) = 0$ for all $\boldsymbol{x}$ we conclude that B must take the form $\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$. □

4.3.2. Shock and Rankine-Hugoniot conditions

In the study of shocks, we need to derive equations for the evolution of $\int_{\Omega_t} C(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}$ where C is smooth everywhere in Ω_t except along a surface Σ_t dividing Ω_t into two parts. By application of Gauss's Theorem in the two parts separately, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} C\,\mathrm{d}\boldsymbol{x} = \int_{\Omega_t}\left[\frac{\partial C}{\partial t} + \mathrm{div}\,(C\,\boldsymbol{v})\right]\mathrm{d}\boldsymbol{x} + \int_{\Sigma_t}[C\,\boldsymbol{V}]\cdot\boldsymbol{n}\,\mathrm{d}S \tag{117}$$

where $\boldsymbol{V} := \boldsymbol{v} - \boldsymbol{w}$ with $\boldsymbol{w}$ the velocity of the surface Σ_t, and $[\cdot]$ denotes the "jump" across the surface.

Now if we assume the evolution of C satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} C\,\mathrm{d}\boldsymbol{x} = \int_{\Omega} f\,\mathrm{d}\boldsymbol{x} \tag{118}$$

for all possible Ω_t's, then we conclude

$$[C\,\boldsymbol{V}]\cdot\boldsymbol{n} = 0 \tag{119}$$

everywhere along Σ_t. (119) is called Rankine-Hugoniot condition for shocks. This condition is usally used to determine the movement of Σ_t.

Example 18. (TRAFFIC FLOW) A simplist, qualitative model for traffic flow is obtained by considering the conservation of mass (numbers of cars):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} \rho(x,t)\,\mathrm{d}x = 0 \tag{120}$$

where $a(t), b(t)$ are the positions of the first and the last of the group of cars under consideration.

Now consider the case of traffic jam. In this case there is a position $x_0(t)$ such that $\rho(x,t) \sim \rho_R$ for $x > x_0(t)$ and $\rho(x,t) \sim \rho_L$ for $x < x_0(t)$ with $\rho_R > \rho_L$. The Rankine-Hugoniot condition (119) now gives

$$\rho_R\,[v_R - x_0'(t)] = \rho_L[v_L - x_0'(t)] \Longrightarrow x_0'(t) = \frac{\rho_R\,v_R - \rho_L\,v_L}{\rho_R - \rho_L}. \tag{121}$$

Once we assume a "constitutive relation" between ρ and v, this would give us an ODE for $x_0'(t)$ which can then be solved to obtain $x_0(t)$ for all t.

5. More Exercises and Problems