# Math 317 Week 08: Integration on Surfaces

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# 1. SURFACES IN $\mathbb{R}^3$

In this and the following sections we restrict ourselves to surfaces in $\mathbb{R}^3$ to avoid cumbersome notations and some technical difficulties.

## 1.1. Parametrization of surfaces

DEFINITION 1. ($C^1$ SURFACE) *Let $D \subseteq \mathbb{R}^2$ be compact and such that $\partial D$ consists of finitely many segments of $C^1$ curves. Assume that*

$$\boldsymbol{r}: G \mapsto \mathbb{R}^3, \qquad \boldsymbol{r}(u,v) := \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} \tag{1}$$

*where $G \supset D$ is open. Let $\boldsymbol{r}$ be $C^1$ on $G$, and one-to-one in $D^o$. Furthermore assume $\left|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right| \neq 0$ in $D^o$. Then we say $S := \boldsymbol{r}(D)$ is a (piece of) $C^1$ surface in $\mathbb{R}^3$.*

**Remark 2.** Notice that $\boldsymbol{r}$ does not need to be one-to-one along the boundary $\partial D$. This is in contrast with the definitions given in Differential Geometry books. The reason is that we are focusing on the integration of continuous functions on such surfaces. In this context $\boldsymbol{r}$ being not one-to-one on $\partial D$ does no harm. This is no longer true when we consider other properties of the surfaces.

**Exercise 1.** Let $\mu$ denote the Jordan measure in $\mathbb{R}^2$. Prove that $\mu(\partial D) = 0$. (Hint:[1] )

**Remark 3.** We will study integration on surfaces that can be written as a union of finitely many pieces of $C^1$ surfaces. We will simply call such surfaces "$C^1$ surfaces". In the following we always assume the surfaces under consideration are $C^1$.

**Remark 4.** We restrict ourselves to $C^1$ surfaces as theory about surfaces that are not $C^1$ can be very involved. Such surfaces are studied in Geometric Measure Theory.

NOTATION 5. *In the following, when no confusion shall arise, we will use $\boldsymbol{r}_u, \boldsymbol{r}_v$ to denote $\frac{\partial \boldsymbol{r}}{\partial u}, \frac{\partial \boldsymbol{r}}{\partial v}$.*

**Remark 6.** Recall that $\boldsymbol{r}_u, \boldsymbol{r}_v$ are tangent vectors to the surface, and thus belong to the plane that is tangent to the surface at that point. The condition $|\boldsymbol{r}_u \times \boldsymbol{r}_v| \neq 0$ guarantees that the two vectors are linearly independent and span the plane, which is consequently unique.

**Exercise 2.** Prove that $\boldsymbol{r}_u \times \boldsymbol{r}_v$ is perpendicular to the surface. (Hint:[2] )

**Exercise 3.** Write down the equation for the tangent plane at some $\boldsymbol{x}_0$ on the surface. (Hint:[3] )

**Example 7.** The sphere

$$\boldsymbol{r}(\phi,\psi) = \begin{pmatrix} R\cos\phi\cos\psi \\ R\sin\phi\cos\psi \\ R\sin\psi \end{pmatrix}, \qquad D = \left\{(\phi,\psi) \,|\, 0 \leqslant \phi \leqslant 2\pi, -\frac{\pi}{2} \leqslant \psi \leqslant \frac{\pi}{2}\right\} \tag{2}$$

1. $D$ is Jordan measurable $\Longleftrightarrow \mu(\partial D) = 0$.
2. That is $\boldsymbol{r}_u \times \boldsymbol{r}_v$ is perpendicular to both $\boldsymbol{r}_u, \boldsymbol{r}_v$.
3. A plane with normal vector $\boldsymbol{N}$ and passes $\boldsymbol{x}_0$ is given by $\boldsymbol{N} \cdot (\boldsymbol{x} - \boldsymbol{x}_0) = 0$.

is one piece of $C^1$-surface.

Another parametrization of the sphere is

$$\boldsymbol{r}(\phi,\psi)=\begin{pmatrix} R\cos\phi\sin\psi \\ R\sin\phi\sin\psi \\ R\cos\psi \end{pmatrix}, \qquad D=\{(\phi,\psi)|\, 0\leqslant\phi\leqslant 2\,\pi, 0\leqslant\psi\leqslant\pi\} \tag{3}$$

Note that in the first parametrization $\psi$ represents the (signed) angle between the vector from the origin to the point and the $x$-$y$ plane, while in the second parametrization it represents the angle between the same vector and the positive $z$-axis.

**Exercise 4.** Suppose you have obtained some formula in spherical coordinates (3) and then decide to switch to (2). Is it possible to "translate" the formula without re-deriving it? (Hint:[4] )

**Exercise 5.** Try to find $\boldsymbol{r}$ for other everyday surfaces, such as the cylinder and the torus.

LEMMA 8. *Let $S$ be a $C^1$ surface. Then for any $\delta>0$ there is $n\in\mathbb{N}$ and $S_1,\ldots,S_n$, all $C^1$ surfaces, such that $S_i^o\cap S_j^o=\varnothing$, and furthermore $d(S_i)<\delta$. Here*

$$d(E):=\sup_{\boldsymbol{x},\boldsymbol{y}\in E}\|\boldsymbol{x}-\boldsymbol{y}\|. \tag{4}$$

**Exercise 6.** Prove the above lemma. (Hint:[5] )

## 1.2. Area of surfaces

### 1.2.1. Area of polyhedron

DEFINITION 9. (POLYHEDRON) *A polyhedron is a bounded set such that its boundary consists of finitely many polygons.*

It is clear that the boundary (surface) of a polyhedron consists of finitely many $C^1$ surfaces, each given by a linear (more precisely, affine) mapping.

To study the area of polygons in $\mathbb{R}^3$, we need some vector algebra. Recall that for $\boldsymbol{u}=\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\boldsymbol{v}=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\in\mathbb{R}^3$, we can define the inner and cross-products as:

$$\boldsymbol{u}\cdot\boldsymbol{v}:=u_1\,v_1+u_2\,v_2+u_3\,v_3; \qquad \boldsymbol{u}\times\boldsymbol{v}:=\begin{pmatrix} u_2\,v_3-u_3\,v_2 \\ u_3\,v_1-u_1\,v_3 \\ u_1\,v_2-u_2\,v_1 \end{pmatrix}. \tag{5}$$

**Exercise 7.** Let $\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^3$. Then

$$\boldsymbol{u}\times\boldsymbol{v}=-\boldsymbol{v}\times\boldsymbol{u}; \tag{6}$$

Further if $a,b,c,d\in\mathbb{R}$, then

$$(a\,\boldsymbol{u}+b\,\boldsymbol{v})\times(c\,\boldsymbol{u}+d\,\boldsymbol{v})=(a\,d-b\,c)\,(\boldsymbol{u}\times\boldsymbol{v}). \tag{7}$$

4. Find the relation between the $\psi$'s in the two formulas.

5. When $S$ is one piece, simply divide $D$. Then use the fact that $\boldsymbol{r}$ is $C^1$ which means $\|\boldsymbol{r}(\boldsymbol{u})-\boldsymbol{r}(\boldsymbol{v})\|\leqslant M\,\|\boldsymbol{u}-\boldsymbol{v}\|$ for some $M>0$.

**Exercise 8.** Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$. Then

$$(\boldsymbol{u} \cdot \boldsymbol{v})^2 + \|\boldsymbol{u} \times \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2. \tag{8}$$

LEMMA 10. *Let $S$ be a triangle in $\mathbb{R}^3$ with vertices $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$. Denote*

$$\boldsymbol{u} := \boldsymbol{x}_3 - \boldsymbol{x}_1, \qquad \boldsymbol{v} := \boldsymbol{x}_2 - \boldsymbol{x}_1. \tag{9}$$

*Then the area of $S$ is given by $\frac{1}{2} \|\boldsymbol{u} \times \boldsymbol{v}\|$.*

**Proof.** Wlog we can assume $\boldsymbol{x}_1 = \mathbf{0}$. Recalling that

$$\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u} \cdot \boldsymbol{v}\|} = \cos\theta \tag{10}$$

where $\theta$ is the angle between $\boldsymbol{u}, \boldsymbol{v}$, by (8) we have

$$\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\| \|\boldsymbol{v}\| \, |\sin(\theta)|. \tag{11}$$

Thus the area of the triangle formed by $\mathbf{0}, \boldsymbol{u}, \boldsymbol{v}$ is given by $\frac{1}{2} \|\boldsymbol{u} \times \boldsymbol{v}\|$. $\square$

COROLLARY 11. *The area of the triangle in $\mathbb{R}^2$ with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is given by*

$$|(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)|. \tag{12}$$

**Exercise 9.** Prove Corollary 11. (Hint:[6] )

LEMMA 12. *Let $S = \boldsymbol{r}(D)$ be a piece of $C^1$ surface given by a linear function*

$$\boldsymbol{r}: G \mapsto \mathbb{R}^3, \qquad \boldsymbol{r}(u, v) := \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}. \tag{13}$$

*Then the surface area of $S$ is*

$$A(S) = \mu(D) \, \|\boldsymbol{r}_u \times \boldsymbol{r}_v\|. \tag{14}$$

*Here $\mu(D)$ is the ($\mathbb{R}^2$) Jordan measure of $D$.*

**Proof.** Since $\boldsymbol{r}$ is linear, we denote

$$\boldsymbol{r}(u, v) = u\,\boldsymbol{a} + v\,\boldsymbol{b} + \boldsymbol{c} \tag{15}$$

where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$. Without loss of generality, we can assume $\boldsymbol{c} = \mathbf{0}$.

We first prove the lemma for the case where $D$ is a triangle with vertices $(0, 0), (u_1, v_1), (u_2, v_2)$. Thus we have

$$\begin{aligned} A(S) &= \frac{1}{2} \|(u_1\,\boldsymbol{a} + v_1\,\boldsymbol{b}) \times (u_2\,\boldsymbol{a} + v_2\,\boldsymbol{b})\| \\ &= \frac{1}{2} \|(u_1 v_2 - v_1 u_2)(\boldsymbol{a} \times \boldsymbol{b})\| \\ &= \frac{|u_1 v_2 - u_2 v_1|}{2} \|\boldsymbol{a} \times \boldsymbol{b}\| = \mu(D) \, \|\boldsymbol{r}_u \times \boldsymbol{r}_v\|. \end{aligned} \tag{16}$$

6. Consider the triangle with vertices $(x_i, y_i, 0)$.

It is clear now that (14) holds when $D$ is a polygon (thus $S$ is also a polygon).

Finally we consider the case $D$ being an arbitrary Jordan measurable set. By properties of such sets we know there are polygons $U_n, V_n$ such that $V_n \subseteq D \subseteq U_n$ and furthermore

$$\lim_{n\rightarrow\infty} \mu(V_n) = \lim_{n\rightarrow\infty} \mu(U_n) = \mu(D). \tag{17}$$

On the other hand, $\boldsymbol{r}(V_n) \subseteq S \subseteq \boldsymbol{r}(U_n)$ and both $\boldsymbol{r}(V_n), \boldsymbol{r}(U_n)$ are polygons too. Therefore $S$ is Jordan measurable with

$$A(S) = \mu(S) = \lim_{n\rightarrow\infty} \mu(\boldsymbol{r}(V_n)) = \lim_{n\rightarrow\infty} \mu(\boldsymbol{r}(U_n)) = \mu(D)\, \|\boldsymbol{r}_u \times \boldsymbol{r}_v\|. \tag{18}$$

Thus ends the proof. □

**Exercise 10.** Prove the formula directly when $D$ is a general triangle.

***What is the above about.***

When a surface $S$ is contained in a plane, there is already a natural way of defining its area through the $\mathbb{R}^2$ Jordan measure – we identify this plane with $\mathbb{R}^2$ and then study its area. Therefore we try to make this our starting point of defining area for more complicated surfaces.

In many books, Definition 13 below is used as the starting point. In that case the above calculation should still be carried out to make sure that this definition does not contradict well-established facts such as the area of a surface contained in the plane should be the same as its $\mathbb{R}^2$ Jordan measure.

### 1.2.2. Area of $C^1$ surfaces

We have unambiguously defined the surface area of polyhedra in $\mathbb{R}^3$ as well as the area for those sets that are contained in a plane. Surprisingly, it turns out that defining area of general surfaces through approximation by polyhedra is not straightforward.[7] For now we will brush the subtlety under the carpet and define area of $C^1$ surfaces through approximating tangent triangles. The derivation is a bit technical and relegated to §4.3, here we simply state the conclusion in the form of a definition.

DEFINITION 13. *Let $S$ be a $C^1$ surface given by $\boldsymbol{r}\colon D \mapsto \mathbb{R}^3$. We define its surface area to be $\int_D \|\boldsymbol{r}_u \times \boldsymbol{r}_v\|\, \mathrm{d}(u, v)$.*

**Remark 14.** In light of Theorem 45, the above definition makes sense. However note that it is an awkward definition compared to arc length since the parametrization is explicitly involved in the definition. Such issues will be settled in Geometric Measure Theory.

COROLLARY 15. *Assume the surface is given by $z = \phi(x, y)$ on $D \subset \mathbb{R}^2$. Then*

$$S = \int_D \sqrt{1 + \phi_x^2 + \phi_y^2}\, \mathrm{d}(x, y). \tag{19}$$

7. See §4.2.

**Exercise 11.** Prove (19).

**Example 16.** Find the area of the part of $z = x\,y$ that is inside $x^2 + y^2 = 1$.

**Solution.** We calculate

$$S = \int_{x^2+y^2 \leqslant 1} \sqrt{1 + z_x^2 + z_y^2}\,\mathrm{d}(x, y) = \frac{2\pi}{3}\,(2\sqrt{2} - 1). \tag{20}$$

***Replacing cross product by dot product.***

Calculating $\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|$ may be unpleasant due to the cross-product involved. It is possible to avoid this and instead calculate three dot products.

THEOREM 17. *Let $S$ be a $C^1$ surface given by $\boldsymbol{r}: D \mapsto \mathbb{R}^3$. If we denote*

$$E := \boldsymbol{r}_u \cdot \boldsymbol{r}_u, \quad F := \boldsymbol{r}_u \cdot \boldsymbol{r}_v, \quad G := \boldsymbol{r}_v \cdot \boldsymbol{r}_v, \tag{21}$$

*then*

$$S = \int_D \sqrt{E\,G - F^2}\,\mathrm{d}(u, v). \tag{22}$$

**Exercise 12.** Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$. Denote

$$E := \boldsymbol{x} \cdot \boldsymbol{x}, \quad F := \boldsymbol{x} \cdot \boldsymbol{y}, \quad G := \boldsymbol{y} \cdot \boldsymbol{y}. \tag{23}$$

Then

$$\|\boldsymbol{x} \times \boldsymbol{y}\| = \sqrt{E\,G - F^2} = \left[\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}\right]^{1/2}. \tag{24}$$

Why does $E\,G - F^2 \geqslant 0$?

**Exercise 13.** Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^3$. Prove $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = \det\begin{pmatrix} \boldsymbol{a}\cdot\boldsymbol{c} & \boldsymbol{a}\cdot\boldsymbol{d} \\ \boldsymbol{b}\cdot\boldsymbol{c} & \boldsymbol{b}\cdot\boldsymbol{d} \end{pmatrix}$.

The bilinear form

$$I(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{x}^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} \boldsymbol{y} \tag{25}$$

is called the "first fundamental form" in the theory of surfaces.

**Exercise 14.** Let $S$ be a $C^1$ surface given by $\boldsymbol{r}: D \mapsto \mathbb{R}^3$ and let $E := \boldsymbol{r}_u \cdot \boldsymbol{r}_u$, $F := \boldsymbol{r}_u \cdot \boldsymbol{r}_v$, $G := \boldsymbol{r}_v \cdot \boldsymbol{r}_v$. Let $\boldsymbol{x}(t) := (u(t), v(t)) \subset D$, $t \in [a, b]$ be a $C^1$ curve in $D$. Prove that $\boldsymbol{r}(u(t), v(t))$ is a $C^1$ curve in $S$, and furthermore its arc length is given by

$$\int_a^b I(\boldsymbol{x}'(t), \boldsymbol{x}'(t))\,\mathrm{d}t = \int_a^b E\,(u')^2 + 2\,F u' v' + G\,(v')^2\,\mathrm{d}t. \tag{26}$$

**Example 18.** Find the surface area of the sphere $x^2 + y^2 + z^2 = R^2$.

**Solution.** We use the parametrization

$$\boldsymbol{r}(\phi, \psi) = \begin{pmatrix} R\cos\phi\cos\psi \\ R\sin\phi\cos\psi \\ R\sin\psi \end{pmatrix}, \qquad D = \left\{(\phi, \psi) \,|\, 0 \leqslant \phi \leqslant 2\pi, -\frac{\pi}{2} \leqslant \psi \leqslant \frac{\pi}{2}\right\}. \tag{27}$$

Then calculate

$$\boldsymbol{r}_u = \begin{pmatrix} -R\sin\phi\cos\psi \\ R\cos\phi\cos\psi \\ 0 \end{pmatrix}, \qquad \boldsymbol{r}_v = \begin{pmatrix} -R\cos\phi\sin\psi \\ -R\sin\phi\sin\psi \\ R\cos\psi \end{pmatrix}. \tag{28}$$

This gives

$$E = R^2(\cos\psi)^2, \qquad F = 0, \qquad G = R^2 \tag{29}$$

and therefore

$$S = \int_D R^2\cos\psi\, \mathrm{d}(\phi,\psi) = 4\pi R^2. \tag{30}$$

**Example 19.** Find the surface area of the torus

$$\boldsymbol{r}(u,v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u), \quad u, v \in (0, 2\pi). \tag{31}$$

**Solution.** We have

$$E = r^2, \qquad F = 0, \qquad G = (r\cos u + a)^2. \tag{32}$$

Therefore calculation gives

$$S = 4\pi^2 r\, a. \tag{33}$$

***Intuitions about the surface area formula.***

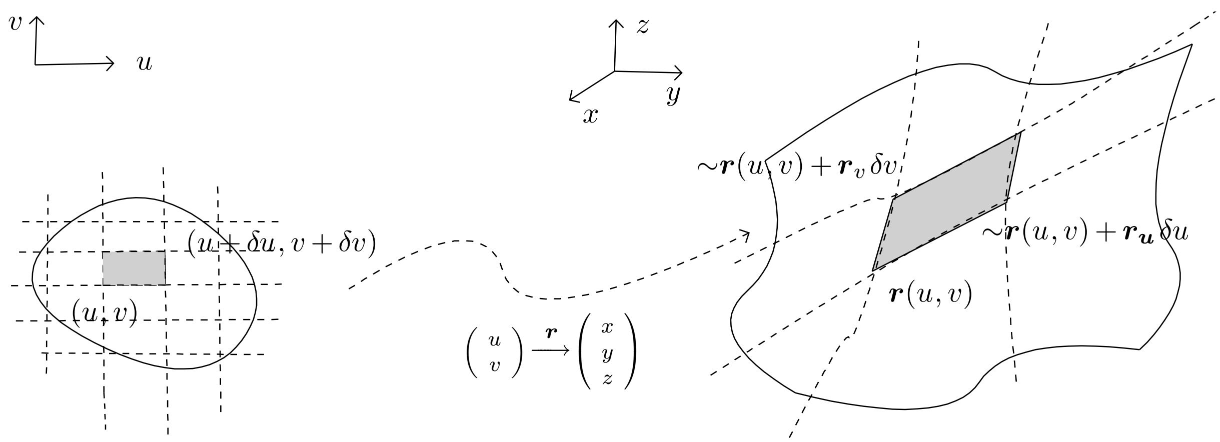


**Figure 1.** Stretching and twisting of of infinitesimal rectangles.

The shaded rectangle in the $(u,v)$-plane, with area $\delta u \cdot \delta v$, is "stretched" by the mapping $\boldsymbol{r}$ to the shaded curvilinear parallelogram in the $(x, y, z)$-space. The sides of this parallelogram are approximately $\boldsymbol{r}_u\,\delta u$ and $\boldsymbol{r}_v\,\delta v$, giving its area to be about $\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|\,\delta u \cdot \delta v$. Summing the areas of all such curvilinear parallelograms up we reach the integral formula

$$\int_D \|\boldsymbol{r}_u \times \boldsymbol{r}_v\|\, \mathrm{d}(u,v). \tag{34}$$

# 2. Surface Integral of Scalar Functions

## 2.1. Definitions and properties

Definition 20. *Let $S$ be a $C^1$ surface. Let $P=\{S_1,\ldots,S_n\}$ be a partition of $S$ into $n$ pieces of $C^1$ surfaces. Let $f: S\mapsto\mathbb{R}$. Denote by $A(S_i)$ the area of $S_i$ for $i=1,2,\ldots,n$. We define the upper and lower sums:*

$$U(f,P):=\sum_{i=1}^n\left[\sup_{\boldsymbol{x}\in S_i}f(\boldsymbol{x})\right]A(S_i);\qquad L(f,P):=\sum_{i=1}^n\left[\inf_{\boldsymbol{x}\in S_i}f(\boldsymbol{x})\right]A(S_i). \tag{35}$$

*Now we define the upper and lower integrals:*

$$U(f):=\inf_P U(f,P);\qquad L(f):=\sup_P L(f,P). \tag{36}$$

*We say $f$ is integrable on $S$ if $U(f)=L(f)$. Further write the common value as*

$$\int_S f(\boldsymbol{x})\,\mathrm{d}S. \tag{37}$$

**Remark 21.** We can also define the integral through Riemann sum.

Theorem 22. *Let $S$ be a $C^1$ surface parametrized by $\boldsymbol{r}: G\mapsto\mathbb{R}^3$. Let $f(x,y,z)$ be continuous on $S$. Then $f$ is integrable on $S$ and*

$$\int_S f(x,y,z)\,\mathrm{d}S=\int_D f(x(u,v),y(u,v),z(u,v))\,\|\boldsymbol{r}_u\times\boldsymbol{r}_v\|\,\mathrm{d}(u,v). \tag{38}$$

*If we denote*

$$E:=\boldsymbol{r}_u\cdot\boldsymbol{r}_u,\quad F:=\boldsymbol{r}_u\cdot\boldsymbol{r}_v,\quad G:=\boldsymbol{r}_v\cdot\boldsymbol{r}_v, \tag{39}$$

*we also have*

$$\int_S f(x,y,z)\,\mathrm{d}S=\int_D f(x(u,v),y(u,v),z(u,v))\sqrt{EG-F^2}\,\mathrm{d}(u,v) \tag{40}$$

**Proof.** The proof is standard and is left as exercise. □

**Exercise 15.** Let $f,g$ be integrable on $S$ and $a,b\in\mathbb{R}$. Then $a\,f+b\,g$ is also integrable on $S$. Furthermore

$$\int_S(a\,f+b\,g)\,\mathrm{d}S=a\int_S f\,\mathrm{d}S+b\int g\,\mathrm{d}S. \tag{41}$$

COROLLARY 23. *When $S$ is given by*

$$z=\phi(x,y), \qquad (x,y)\in D, \tag{42}$$

*then*

$$\int_S f(x,y,z)\,\mathrm{d}S=\int_D f(x,y,\phi(x,y))\,\sqrt{1+\phi_x^2+\phi_y^2}\,\mathrm{d}(x,y). \tag{43}$$

**Exercise 16.** Prove (43).

**Remark 24.** Note that the above results clearly still hold for surfaces $S=S_1\cup\cdots\cup S_m$ where each $S_i$ is a $C^1$ surface and $i\neq j\Longrightarrow S_i^o\cap S_j^o=\varnothing$.

## 2.2. Calculations

***Surface integral of the first type: Formulas.***

- When $S$ is given by $(x(u,v),y(u,v),z(u,v))$.

$$\begin{aligned}\int_S f(x,y,z)\,\mathrm{d}S &= \int_D f(x(u,v),y(u,v),z(u,v))\,\|\boldsymbol{r}_u\times\boldsymbol{r}_v\|\,\mathrm{d}(u,v)\\ &= \int_D f(x(u,v),y(u,v),z(u,v))\,\sqrt{EG-F^2}\,\mathrm{d}(u,v).\end{aligned} \tag{44}$$

  Here $E:=\boldsymbol{r}_u\cdot\boldsymbol{r}_u, \quad F:=\boldsymbol{r}_u\cdot\boldsymbol{r}_v, \quad G:=\boldsymbol{r}_v\cdot\boldsymbol{r}_v$.

- When $S$ is given by $z=\phi(x,y)$.

$$\int_S f(x,y,z)\,\mathrm{d}S=\int_D f(x,y,\phi(x,y))\,\sqrt{1+\phi_x^2+\phi_y^2}\,\mathrm{d}(x,y). \tag{45}$$

**Example 25.** Calculate

$$\int_S z^2\,\mathrm{d}S \tag{46}$$

where $S:=\{(x,y,z)|\,x^2+y^2+z^2=R^2\}$.

**Solution.** We parametrize $S$:

$$\boldsymbol{r}(\phi,\psi)=\begin{pmatrix}R\cos\phi\cos\psi\\ R\sin\phi\cos\psi\\ R\sin\psi\end{pmatrix}, \qquad D=\left\{(\phi,\psi)|\,0\leqslant\phi\leqslant 2\pi,-\frac{\pi}{2}\leqslant\psi\leqslant\frac{\pi}{2}\right\}. \tag{47}$$

Then we have

$$\boldsymbol{r}_\phi=R\begin{pmatrix}-\sin\phi\cos\psi\\ \cos\phi\cos\psi\\ 0\end{pmatrix}, \qquad \boldsymbol{r}_\psi=R\begin{pmatrix}-\cos\phi\sin\psi\\ -\sin\phi\sin\psi\\ \cos\psi\end{pmatrix} \tag{48}$$

and

$$E = R^2 \cos^2\psi, \quad F = 0, \quad G = R^2 \Longrightarrow \sqrt{E F - G^2} = R^2\,|\cos\psi| = R^2 \cos\psi. \tag{49}$$

Note that the last equality follows from the fact that $-\frac{\pi}{2} \leqslant \psi \leqslant \frac{\pi}{2}$.

Now we calculate

$$\begin{aligned} I &= \int_D (R\sin\psi)^2 R^2 \cos\psi \\ &= R^4 \int_D (\sin\psi)^2 \cos\psi\,\mathrm{d}\psi \\ &= R^4 \int_0^{2\pi} \left[\int_{-\pi/2}^{\pi/2} (\sin\psi)^2 \cos\psi\,\mathrm{d}\psi\right] \mathrm{d}\phi \\ &= R^4 \int_0^{2\pi} \left[\frac{(\sin\psi)^3}{3}\right]_{-\pi/2}^{\pi/2} \mathrm{d}\phi \\ &= R^4 \int_0^{2\pi} \frac{2}{3}\,\mathrm{d}\phi = \frac{4\pi R^4}{3}. \end{aligned} \tag{50}$$

**Remark 26.** The calculation can be much simplified through the following trick:

By symmetry we see that

$$\int_S z^2\,\mathrm{d}S = \frac{1}{3}\int_S (x^2 + y^2 + z^2)\,\mathrm{d}S = \frac{1}{3}\int_S \mathrm{d}S = \frac{4\pi}{3}. \tag{51}$$

**Example 27.** Calculate

$$\int_S \frac{1}{z}\,\mathrm{d}S \tag{52}$$

where $S := \{(x, y, z)|\, x^2 + y^2 + z^2 = 1\} \cap \{(x, y, z)|\, z \geqslant h\}$.

**Solution.** We write

$$S\colon \quad z = \phi(x, y) := \sqrt{1 - x^2 - y^2}, \qquad D\colon \quad x^2 + y^2 \leqslant 1 - h^2. \tag{53}$$

This leads to

$$\sqrt{1 + \phi_x^2 + \phi_y^2} = \frac{1}{\sqrt{1 - x^2 - y^2}}. \tag{54}$$

Then we have

$$\begin{aligned} \int_S \frac{\mathrm{d}S}{z} &= \int_D \frac{1}{1 - (x^2 + y^2)}\,\mathrm{d}(x, y) \\ &= \int_0^{2\pi} \left[\int_0^{\sqrt{1-h^2}} \frac{1}{1 - r^2}\, r\,\mathrm{d}r\right] \mathrm{d}\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{2}\ln|1 - r^2|\right]_0^{\sqrt{1-h^2}} \mathrm{d}\theta \\ &= -2\pi\ln h = 2\pi\ln(1/h). \end{aligned} \tag{55}$$

Alternatively we could parametrize using spherical coordinates.

**Solution.** If we parametrize using spherical coordinates, then we have

$$\boldsymbol{r}(\phi,\psi)=\begin{pmatrix}\cos\phi\cos\psi\\ \sin\phi\cos\psi\\ \sin\psi\end{pmatrix},\qquad D=\{(\phi,\psi)|\,0\leqslant\phi\leqslant 2\,\pi,\arcsin h\leqslant\psi\leqslant\pi/2\}. \tag{56}$$

Then

$$\begin{aligned}\int_S\frac{\mathrm{d}S}{z} &= \int_D\frac{1}{\sin\psi}\cos\psi\,\mathrm{d}(\phi,\psi)\\ &= 2\,\pi\int_{\arcsin h}^{\pi/2}\frac{\cos\psi}{\sin\psi}\,\mathrm{d}\psi\\ (u:=\sin\psi)\quad &= 2\,\pi\int_h^1\frac{\mathrm{d}u}{u}\\ &= 2\,\pi\ln(1/h).\end{aligned}\tag{57}$$

**Example 28.** (DEMIDOVICH 2353) Determine the coordinates of the centre of gravity of a homogeneous parabolic envelope $z=x^2+y^2$ $(0\leqslant z\leqslant 1)$.

**Solution.** Clearly the $x,y$ coordinates are zero, while

$$z_c:=\left(\int_S\mathrm{d}S\right)^{-1}\left(\int_S z\,\mathrm{d}S\right). \tag{58}$$

We parametrize

$$S\colon\quad\begin{pmatrix}x\\ y\\ \phi(x,y):=x^2+y^2\end{pmatrix}\Longrightarrow\sqrt{1+\phi_x^2+\phi_y^2}=\sqrt{1+4\,(x^2+y^2)};\qquad D:=0\leqslant x^2+y^2\leqslant 1. \tag{59}$$

Now

$$\begin{aligned}\int_S\mathrm{d}S &= \int_D\sqrt{1+4\,(x^2+y^2)}\,\mathrm{d}(x,y)\\ &= \int_0^{2\pi}\left[\int_0^1\sqrt{1+4\,r^2}\,r\,\mathrm{d}r\right]\mathrm{d}\theta\\ &= \frac{\pi}{4}\int_0^4\sqrt{1+u}\,\mathrm{d}u=\frac{\pi\left(5\sqrt{5}-1\right)}{6}.\end{aligned}\tag{60}$$

$$\begin{aligned}\int_S z\,\mathrm{d}S &= \int_D(x^2+y^2)\sqrt{1+4\,(x^2+y^2)}\,\mathrm{d}(x,y)\\ &= \int_0^{2\pi}\left[\int_0^1\sqrt{1+4\,r^2}\,r^3\,\mathrm{d}r\right]\mathrm{d}\theta\\ (r=u/2)\quad &= \frac{\pi}{8}\int_0^2\sqrt{1+u^2}\,u^3\,\mathrm{d}u\\ (u:=\sinh x)\quad &= \frac{\pi}{8}\int_0^{x_0}(\cosh x)^2(\sinh x)^3\,\mathrm{d}x\\ &= \frac{\pi}{60}\left(1+25\sqrt{5}\right).\end{aligned}\tag{61}$$

Therefore

$$z_c=\frac{1+25\sqrt{5}}{10\left(5\sqrt{5}-1\right)}. \tag{62}$$

# 3. Surface Integral of Vector Functions

## 3.1. Orientation of surfaces

Definition 29. (Orientable surface) *A $C^1$ surface $S$ is **orientable** if and only if there is a vector valued function*

$$\boldsymbol{N}: S \mapsto \mathbb{R}^3 \tag{63}$$

*that is continuous, normal to $S$, and of unit length at every $(x, y, z) \in S$. Otherwise $S$ is said to be **nonorientable**.*

**Remark 30.** Note that as $\boldsymbol{N}$ is only required to be continuous, the above definition still applies in the case $S$ is a finite union of $C^1$ surfaces.

**Example 31.** Let $S$ be given by $z = \phi(x, y)$, $(x, y) \in D \subseteq \mathbb{R}^2$. Then $S$ is orientable, with

$$\boldsymbol{N}(x, y, \phi(x, y)) = \frac{1}{\sqrt{1 + \phi_x^2 + \phi_y^2}} \begin{pmatrix} -\phi_x \\ -\phi_y \\ 1 \end{pmatrix} \quad \text{or} \quad -\frac{1}{\sqrt{1 + \phi_x^2 + \phi_y^2}} \begin{pmatrix} -\phi_x \\ -\phi_y \\ 1 \end{pmatrix}. \tag{64}$$

Note that it's either $+$ everywhere or $-$ everywhere.

**Example 32.** The sphere $\{(x, y, z) | \, x^2 + y^2 + z^2 = R^2\}$ is orientable, with

$$\boldsymbol{N}(x, y, z) = \frac{1}{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{or} \qquad -\frac{1}{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{65}$$

**Example 33.** (Möbius band) Consider

$$D := \{(u, v) | \, -1 \leqslant u \leqslant 1, 0 \leqslant v \leqslant 2\pi\}, \tag{66}$$

$$\boldsymbol{r}(u, v) := \begin{pmatrix} \left(2 + u \sin\left(\frac{v}{2}\right)\right) \cos v \\ \left(2 + u \sin\left(\frac{v}{2}\right)\right) \sin v \\ u \cos\left(\frac{v}{2}\right) \end{pmatrix}. \tag{67}$$

This surface is not orientable.

Lemma 34. *Let $D \subseteq \mathbb{R}^3$ be compact. We say $D$ is a regular region if $D = \overline{D^o}$, that is $D$ is the closure of its interior. Assume $S = \partial D$ is $C^1$. Then $S$ is orientable.*

**Proof.** See §4.1.1. □

**Remark 35.** The natural orientation of such $S$ is the "outer normal", that is the normal vector that is pointing in to $D^c$. In other words there is $\delta > 0$ such that $\{\boldsymbol{x}_0 + t\, \boldsymbol{n}(\boldsymbol{x}_0)\} \subset D^c$ for all $t \in (0, \delta)$.

Lemma 36. *Let $S$ be one piece of $C^1$ surface parametrized by $\boldsymbol{r}: G \mapsto \mathbb{R}^3$ satisfying $\boldsymbol{r}_u \times \boldsymbol{r}_v \neq \boldsymbol{0}$ everywhere on $S$. Then if $S$ is orientable, on each connected component of $S$ there must holds either*

$$\boldsymbol{N}(x, y, z) = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|} \qquad \forall (x, y, z) \in S \tag{68}$$

*or*

$$\boldsymbol{N}(x, y, z) = -\frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|} \qquad \forall (x, y, z) \in S. \tag{69}$$

**Proof.** See 4.1.2. □

**Exercise 17.** Try to apply the above lemma to Möbius band.

## 3.2. Definitions and properties

DEFINITION 37. (SURFACE INTEGRAL OF VECTOR FIELDS) *Let $S$ be an orientable $C^1$ surface with orientation given by a normal vector field $\boldsymbol{n}$. Let $\boldsymbol{f} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}: S \mapsto \mathbb{R}^3$ be continuous. Then we define*

$$\int_S \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{S} := \int_S f(x, y, z)\, \mathrm{d}y \wedge \mathrm{d}z + g(x, y, z)\, \mathrm{d}z \wedge \mathrm{d}x + h(x, y, z)\, \mathrm{d}x \wedge \mathrm{d}y := \int_S \boldsymbol{f} \cdot \boldsymbol{n}\, \mathrm{d}S. \tag{70}$$

***What does $\mathbf{d}y \wedge \mathbf{d}z$ mean?***

For now it means nothing. $\mathrm{d}y \wedge \mathrm{d}z$ is just a symbol. On the other hand we can intuitively interpret

$$\begin{pmatrix} \mathrm{d}y \wedge \mathrm{d}z \\ \mathrm{d}z \wedge \mathrm{d}x \\ \mathrm{d}x \wedge \mathrm{d}y \end{pmatrix} \text{ as } \boldsymbol{n}\, \mathrm{d}S = \begin{pmatrix} n_x\, \mathrm{d}S \\ n_y\, \mathrm{d}S \\ n_z\, \mathrm{d}S \end{pmatrix}. \tag{71}$$

where $n_x, n_y, n_z$ are components of the normal vector $\boldsymbol{n}$.

Precise meanings will be assigned to such symbols in differential geometry. There $f(x, y, z)\, \mathrm{d}y \wedge \mathrm{d}z + g(x, y, z)\, \mathrm{d}z \wedge \mathrm{d}x + h(x, y, z)\, \mathrm{d}x \wedge \mathrm{d}y$ will be interperted as a "two-form", and two-forms can be paired with two-dimensional objects, such as the surface $S$, to yield a number – call it the integral. Similarly, $f\, \mathrm{d}x + g\, \mathrm{d}y + h\, \mathrm{d}z$ is a one-form which is naturally integrated along one-dimensional objects such as curves.

PROPOSITION 38. *Let $S$ be an orientable $C^1$ surface with orientation given by a normal vector field $\boldsymbol{n}$. Let $\boldsymbol{f}_i = \begin{pmatrix} f_i \\ g_i \\ h_i \end{pmatrix}: S \mapsto \mathbb{R}^3, i = 1, 2$ be continuous and $a_1, a_2 \in \mathbb{R}$. Then*

$$\int_S (a_1\, \boldsymbol{f}_1 + a_2\, \boldsymbol{f}_2) \cdot \mathrm{d}\boldsymbol{S} = a_1 \int_S \boldsymbol{f}_1 \cdot \mathrm{d}\boldsymbol{S} + a_2 \int_S \boldsymbol{f}_2 \cdot \mathrm{d}\boldsymbol{S}. \tag{72}$$

*In particular,*

$$\int_S f(x, y, z)\, \mathrm{d}y \wedge \mathrm{d}z + g(x, y, z)\, \mathrm{d}z \wedge \mathrm{d}x + h(x, y, z)\, \mathrm{d}x \wedge \mathrm{d}y \tag{73}$$

*equals*

$$\int_S f(x, y, z)\, \mathrm{d}y \wedge \mathrm{d}z + \int_S g(x, y, z)\, \mathrm{d}z \wedge \mathrm{d}x + \int_S h(x, y, z)\, \mathrm{d}x \wedge \mathrm{d}y. \tag{74}$$

**Proof.** The conclusion follows immediately from (70). □

PROPOSITION 39. *Let $S_i, i = 1, 2$ be orientable $C^1$ surfaces with orientation given by a normal vector field $\boldsymbol{n}_i$. Let $\boldsymbol{f}$ be continuous. Then*

$$\int_{S_1 \cup S_2} \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{S} = \int_{S_1} \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{S} + \int_{S_2} \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{S}. \tag{75}$$

**Proof.** The conclusion follows immediately from (70). □

Theorem 40. *Let a orientable $C^1$ surface $S$ be parametrized by $\boldsymbol{r}\colon D \mapsto \mathbb{R}^3$ consistently with the orientation (the orientation $\boldsymbol{n}$ is given by $\frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|}$. Then*

$$\int_S f\,\mathrm{d}y \wedge \mathrm{d}z + g\,\mathrm{d}z \wedge \mathrm{d}x + h\,\mathrm{d}x \wedge \mathrm{d}y = \int_D \boldsymbol{f}(\boldsymbol{r}(u,v)) \cdot [\boldsymbol{r}_u \times \boldsymbol{r}_v]\,\mathrm{d}(u,v). \tag{76}$$

*This can be further simplified to*

$$\int_D f \cdot \det\left(\frac{\partial(y,z)}{\partial(u,v)}\right) + g \cdot \det\left(\frac{\partial(z,x)}{\partial(u,v)}\right) + h \cdot \det\left(\frac{\partial(x,y)}{\partial(u,v)}\right) \mathrm{d}(u,v) \tag{77}$$

*where $\frac{\partial(y,z)}{\partial(u,v)}$, etc., are the Jacobians.*

**Proof.** We have

$$\int_S \boldsymbol{f} \cdot \boldsymbol{n}\,\mathrm{d}S = \int_D \boldsymbol{f}(\boldsymbol{r}(u,v)) \cdot \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|}\,\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|\,\mathrm{d}(u,v) \tag{78}$$

and the conclusions immediately follow. □

Corollary 41. *When the surface is given by $z = \phi(x, y)$, $(x, y) \in D$ with the normal pointing upward. Then $\int_S f\,\mathrm{d}y \wedge \mathrm{d}z + g\,\mathrm{d}z \wedge \mathrm{d}x + h\,\mathrm{d}x \wedge \mathrm{d}y$ equals*

$$\int_D [-f(x,y,\phi(x,y))\,\phi_x - g(x,y,\phi(x,y))\,\phi_y + h(x,y,\phi(x,y))]\,\mathrm{d}(x,y). \tag{79}$$

**Proof.** In this case we have

$$\boldsymbol{r}(x,y) = \begin{pmatrix} u \\ v \\ \phi(u,v) \end{pmatrix} \Longrightarrow \boldsymbol{r}_u = \begin{pmatrix} 1 \\ 0 \\ \phi_u \end{pmatrix}, \boldsymbol{r}_v = \begin{pmatrix} 0 \\ 1 \\ \phi_v \end{pmatrix} \Longrightarrow \boldsymbol{r}_u \times \boldsymbol{r}_v = \begin{pmatrix} -\phi_u \\ -\phi_v \\ 1 \end{pmatrix} \tag{80}$$

and the conclusion follows. □

## 3.3. Calculations

The calculation of such integrals is actually quite easy.

***Surface integral of the second kind: Formulas.***

- When $S$ is given by $(x(u,v), y(u,v), z(u,v))$ and furthermore the specified normal $\boldsymbol{n}$ points in the same direction as $\boldsymbol{r}_u \times \boldsymbol{r}_v$,

$$\begin{aligned} \int_S \boldsymbol{f}(x,y,z) \cdot \mathbf{d}\boldsymbol{S} &= \int_D \boldsymbol{f}(\boldsymbol{r}(u,v)) \cdot [\boldsymbol{r}_u \times \boldsymbol{r}_v]\,\mathrm{d}(u,v) \\ &= \int_D f \det\frac{\partial(y,z)}{\partial(u,v)} + g \det\frac{\partial(z,x)}{\partial(u,v)} + h \det\frac{\partial(x,y)}{\partial(u,v)} \mathrm{d}(u,v). \end{aligned} \tag{81}$$

  Note that the $f, g, h$ in the above formula are all evaluated at $(x(u,v), y(u,v), z(u,v))$.

- When $S$ is given by $z = \phi(x, y)$ and furthermore the specified normal $\boldsymbol{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$ satisfies $n_z > 0$, that is points upward,

$$\int_S \boldsymbol{f}(x,y,z) \cdot \mathbf{d}\boldsymbol{S} = \int_D [-f(x,y,\phi)\,\phi_x - g(x,y,\phi)\,\phi_y + h(x,y,\phi)]\,\mathrm{d}(x,y). \tag{82}$$

**Example 42.** Calculate

$$I = \int_{S_i} x^2\,\mathrm{d}y \wedge \mathrm{d}z + y^2\,\mathrm{d}z \wedge \mathrm{d}x + z^2\,\mathrm{d}x \wedge \mathrm{d}y \tag{83}$$

where $S_1$ is the triangle with vertices $(1,0,0),(0,1,0),(0,0,1)$, and the orientation is such that the normal is pointing downward; $S_2$ is the sphere $x^2+y^2+z^2=1$, with the normal pointing outward.

**Solution.** We notice that in either case there is symmetry with respect to the permutation $x \to y \to z \to x$, therefore

$$I = 3\int_{S_i} z^2\,\mathrm{d}x \wedge \mathrm{d}y. \tag{84}$$

Thus for $S_1$ we parametrize

$$S_1: \qquad z = \phi(x,y) := 1 - x - y, \qquad D := \{(x,y) | \, 0 \leqslant x, y; x + y \leqslant 1\} \tag{85}$$

Note that the specified $\boldsymbol{n}$ points downward, therefore we should use $\begin{pmatrix} \phi_x \\ \phi_y \\ -1 \end{pmatrix}$ in our calculation.

We have

$$\begin{aligned} I &= 3\int_{S_1} z^2\,\mathrm{d}x \wedge \mathrm{d}y \\ &= 3\int_D (1-x-y)^2\, n_z\,\mathrm{d}(x,y) \\ &= -3\int_D (1-x-y)^2\,\mathrm{d}(x,y) \\ &= -3\int_0^1 \left[\int_0^{1-x} (1-x-y)^2\,\mathrm{d}y\right]\mathrm{d}x \\ &= -3\int_0^1 \left[(1-x)^3 - (1-x)^3 + \frac{1}{3}(1-x)^3\right]\mathrm{d}x \\ &= -\frac{1}{4}. \end{aligned} \tag{86}$$

For $S_2$ we have $I = 0$ thanks to symmetry.

**Example 43.** (FOLLAND) Let $S$ be the portion of the cone $x^2+y^2=z^2$ with $0 \leqslant z \leqslant 1$, oriented so that the normal points upward. Calculate

$$I = \int_S x^2\,\mathrm{d}y \wedge \mathrm{d}z + y\,z\,\mathrm{d}z \wedge \mathrm{d}x + y\,\mathrm{d}x \wedge \mathrm{d}y. \tag{87}$$

**Solution.** We set $D = \{x^2+y^2 \leqslant 1\}$. Then after some calculation we have

$$\begin{aligned} I &= \int_{x^2+y^2 \leqslant 1} \left[\frac{-x^3 - y^2\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} + y\right]\mathrm{d}(x,y) \\ &= \int_0^{2\pi}\int_0^1 [-r^3(\cos\theta)^3 - r^3(\sin\theta)^2 + r^2\sin\theta]\,\mathrm{d}r\,\mathrm{d}\theta \\ &= -\frac{\pi}{4}. \end{aligned}$$

# 4. Advanced Topics, Notes, Comments

## 4.1. Proofs of some theorems

### 4.1.1. Proof of Lemma 34

**Proof.** First we prove that there exists a unique function $\boldsymbol{n}(\boldsymbol{x})\colon S \mapsto S_2$ (the unit sphere in $\mathbb{R}^3$) satisfying

$$\exists \delta > 0, \qquad \forall t \in (0, \delta), \qquad \boldsymbol{x} + t\,\boldsymbol{n}(\boldsymbol{x}) \in D^c. \tag{88}$$

Fix any $\boldsymbol{x}_0 \in S$. Since $S$ is $C^1$, there is a well-defined normal vector $\boldsymbol{n}$ at $\boldsymbol{x}_0$, and furthermore there is $r > 0$ such that $S \cap B(\boldsymbol{x}_0, r)$ is given by some $C^1$ mapping $\boldsymbol{r}(u, v)\colon E \subseteq \mathbb{R}^2 \mapsto \mathbb{R}^3$. Denote $\boldsymbol{x}_0 = \boldsymbol{r}(u_0, v_0)$.

Consider the mapping

$$\boldsymbol{R}(u, v, t) := \boldsymbol{r}(u, v) + t\,\boldsymbol{n}. \tag{89}$$

It is easy to check that the Jacobian of $\boldsymbol{R}$ is non-singular at $(u_0, v_0, 0)$. Thus by Inverse Function Theorem there is $r_1 > 0$ such that $\boldsymbol{R}^{-1}\colon B(\boldsymbol{x}_0, r_1) \mapsto \mathbb{R}^3$ exists and is $C^1$. We denote $V := \boldsymbol{R}^{-1}(B(\boldsymbol{x}_0, r_1))$. Note that $V$ is a connected open set.

We would like to prove $\boldsymbol{R}^{-1}(D^o \cap B(\boldsymbol{x}_0, r_1))$ is either$\{t > 0\} \cap V$ or in $\{t < 0\} \cap V$. Assume otherwise. Then the following are possible situations.

i. Both $\boldsymbol{R}^{-1}(D^o \cap B(\boldsymbol{x}_0, r_1)) \cap \{t > 0\} \cap V$ and $\boldsymbol{R}^{-1}(D^c \cap B(\boldsymbol{x}_0, r_1)) \cap \{t > 0\} \cap V$ are non-empty. Since $\boldsymbol{R}^{-1}(S \cap B(\boldsymbol{x}_0, r_1)) \cap \{t > 0\} \cap V = \varnothing$, the openness of both $\boldsymbol{R}^{-1}(D^o \cap B(\boldsymbol{x}_0, r_1))$ and $\boldsymbol{R}^{-1}(D^c \cap B(\boldsymbol{x}_0, r_1))$ contradict the connectedness of $\{t > 0\} \cap V$.

ii. Both $\boldsymbol{R}^{-1}(D^o \cap B(\boldsymbol{x}_0, r_1)) \cap \{t < 0\} \cap V$ and $\boldsymbol{R}^{-1}(D^c \cap B(\boldsymbol{x}_0, r_1)) \cap \{t < 0\} \cap V$ are non-empty. This leads similarly to contradiction.

iii. $\boldsymbol{R}^{-1}(D^o \cap B(\boldsymbol{x}_0, r_1)) = \{t \neq 0\} \cap V$. Since $D = \overline{D^o}$, this implies $\boldsymbol{R}^{-1}(D \cap B(\boldsymbol{x}_0, r_1)) = V$ which implies $S \subset D^o$. Contradiction.

iv. $\boldsymbol{R}^{-1}(D^c \cap B(\boldsymbol{x}_0, r_1)) = \{t \neq 0\} \cap V$. This leads similarly to contradiction.

Thus we see that $D^o \cap B(\boldsymbol{x}_0, r_1)$ and $D^c \cap B(\boldsymbol{x}_0, r_1)$ are $\boldsymbol{R}(\{t > 0\} \cap V)$ and $\boldsymbol{R}(\{t < 0\} \cap V)$. We can define uniquely $\boldsymbol{n}(\boldsymbol{x}_0)$ to be the one pointing "outward", that is $\boldsymbol{x}_0 + t\,\boldsymbol{n}(\boldsymbol{x}_0) \subset D^c$ for all small positive $t$. In other words, at every $\boldsymbol{x}_0 \in S$, among the two normal vectors $\pm\boldsymbol{n}(\boldsymbol{x})$, exactly one points outward and one points inward.

Now we define $\boldsymbol{n}(\boldsymbol{x})$ to be the outward pointing normal vector and prove that it is continuous. Assume otherwise. Then there is $\boldsymbol{x}_0 \in S$ and $\boldsymbol{x}_n \longrightarrow \boldsymbol{x}_0$ such that $\boldsymbol{n}(\boldsymbol{x}_n) \longrightarrow -\boldsymbol{n}(\boldsymbol{x}_0)$. From the above discussion we know that there is $B(\boldsymbol{x}_0, r_0)$ for some $r_0 > 0$ and a $C^1$ mapping $\boldsymbol{R}\colon B(\boldsymbol{x}_0, r) \mapsto V \subseteq \mathbb{R}^3$ such that $\boldsymbol{R}(D^c \cap B(\boldsymbol{x}_0, r_0)) = V \cap \{t > 0\}$, and $\boldsymbol{R}(\boldsymbol{x}_0 + \boldsymbol{n}(\boldsymbol{x}_0)) = (u_0, v_0, 1)$. Together with the discontinuity assumption we now have

$$\lim_{n \to \infty} \boldsymbol{R}(\boldsymbol{x}_n + \boldsymbol{n}(\boldsymbol{x}_n)) = (u_0, v_0, -1) \tag{90}$$

which contradicts the continuity of $\boldsymbol{R}$. $\square$

### 4.1.2. Proof of Lemma 36

**Proof.** Assume $S$ is orientable. By defintion of $C^1$ surfaces $S$ is connected.

Fix a point $(x_0, y_0, z_0) \in S$. Since $\frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|}(x_0, y_0, z_0)$ is normal to $S$ and has unit length, we must have either $\boldsymbol{N} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|}$ or $\boldsymbol{N} = -\frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|}$ at $(x_0, y_0, z_0)$. Wlog we assume the former case. Now we try to prove that $\boldsymbol{N} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|}$ at all other $(x, y, z) \in S$ too.

We consider $f: S \mapsto \mathbb{R}$ defined through

$$f(x, y, z) = \boldsymbol{N}(x, y, z) \cdot \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|}(x, y, z). \tag{91}$$

Then we have

i. $f(x_0, y_0, z_0) = 1$;

ii. $f(x, y, z) \in \{-1, 1\}$ for all $(x, y, z) \in S$.

Now as $\boldsymbol{r}_u \times \boldsymbol{r}_v \neq \boldsymbol{0}$ everywhere on $S$, $\frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|}$ is continuous on $S$. Together with the continuity assumption on $\boldsymbol{N}$, we conclude that $f(x, y, z)$ is a continuous function on $S$. By the intermediate value theorem, if there is $(x, y, z) \in S$ such that $f(x, y, z) = -1$, then along any curve connecting $(x, y, z)$ and $(x_0, y_0, z_0)$ there is a point $(x', y', z')$ such that $f(x', y', z') = 0$. Contradiction. $\square$

## 4.2. The counterexample by H. A. Schwarz

> "The example of Schwarz, ... , was the starting point of an extensive and fascinating literature. Still, we do not possess as yet a satisfactory theory of the area of surfaces, ..."
>
> —— Tibor Rado, 1943[8]

- Gelbaum, B. R. and Olmsted, J. M. H., *Counterexamples in Analysis*, Chapter 11, Example 7.

Let

$$S = \{(x, y, z) | \, x^2 + y^2 = 1, \quad 0 \leqslant z \leqslant 1\}. \tag{92}$$

Let $m \in \mathbb{N}$. Define $2\,m + 1$ circles:

$$C_{k,m} := S \cap \left\{(x, y, z) | \, z = \frac{k}{2\,m}\right\}, \qquad k = 0, 1, 2, \ldots, 2\,m. \tag{93}$$

Now let $n \in \mathbb{N}$. Pick on each $C_{k,m}$ $n$ points:

$$P_{k,m,j} := \begin{cases} \left(\cos\frac{2\,j\,\pi}{n}, \sin\frac{2\,j\,\pi}{n}, \frac{k}{2\,m}\right) & k \text{ even} \\ \left(\cos\frac{(2\,j+1)\,\pi}{n}, \sin\frac{(2\,j+1)\,\pi}{n}, \frac{k}{2\,m}\right) & k \text{ odd} \end{cases}, \qquad j = 0, 1, \ldots, n-1. \tag{94}$$

Connecting this points in a natural manner we obtain $4\,m\,n$ congruent space triangles. It can be calculated that the area of each triangle is

$$\sin\left(\frac{\pi}{n}\right)\left[\frac{1}{4\,m^2} + \left(1 - \cos\left(\frac{\pi}{n}\right)\right)^2\right]^{1/2}. \tag{95}$$

8. Tibor Rado, *What is the Area of a Surface?*, The American Mathematical Monthly, Vol. 50, No. 3, Mar., 1943, pp. 139 - 141.

**Exercise 18.** Prove the above formula.

Thus the area of the polyhedron inscribed in the cylinder is

$$A_{mn} := 2\,\pi \frac{\sin(\pi/n)}{\pi/n}\left(1 + 4\,m^2\left(1 - \cos\frac{\pi}{n}\right)^2\right)^{1/2}. \tag{96}$$

**Exercise 19.** Prove that, as $m, n \to \infty$,

a) the diameters of the triangles $\longrightarrow 0$;

b) The limit of $A_{mn}$ depends on how $m, n \longrightarrow \infty$. Furthermore for any $s > 2\,\pi$ (including $\infty$), there is a strictly increasing function $M: \mathbb{N} \mapsto \mathbb{N}$ such that

$$\lim_{n \to \infty} A_{M(n),n} = s. \tag{97}$$

Note that the area of the cylinder is $2\,\pi$.

**Remark 44.** (From (Lord) ) In 1868 J. A. Serret[9] suggested the "obvious" generalization of the natural method of finding arc length to calculation of surface area:

> "Given a portion of a curved surface bounded by a curve $C$, we call the area of this surface the limit $S$ towards which the area of an inscribed polyhedral surface tends, where the inscribed polyhedral surface is formed by triangular faces and is bounded by the polygonal curve $G$, which limits the curve $C$"
>
> "One must show that the limit $S$ exists and that it is independent of the way in which the faces of the inscribed surface decreases."

The problem with this approach was first realized by H. A. Schwarz[10], who wrote to Italian mathematician Gennochi about this in 1880. Later in 1882 Gennochi's student Peano annouced the same result in a course he taught. Around the same time Schwarz wrote to Hermite about his example. Hermite published Schwarz's letter in his course notes, which was published later than that of Peano's. Consequently there are disputes about priority.

## 4.3. Area of $C^1$ surfaces

Theorem 45. *Let $S$ be a piece of $C^1$ surface parametrized by $\boldsymbol{r}: D \mapsto \mathbb{R}^3$. Let $P := \{D_1, \ldots, D_n\}$ be any partition of $D$, that is*

$$D = \cup_{k=1}^{n} D_k, \qquad i \neq j \Longrightarrow D_i^o \cap D_j^o = \varnothing, \qquad \forall i, \partial D_i \text{ consists of finitely many } C^1 \text{ curves.} \tag{98}$$

*Let $(u_1, v_1) \in D_1, \ldots, (u_n, v_n) \in D_n$ be arbitrary. Denote*

$$\boldsymbol{r}_i(u, v) := \boldsymbol{r}(u_i, v_i) + \boldsymbol{r}_u\,(u - u_i) + \boldsymbol{r}_v\,(v - v_i), \qquad S_i = \boldsymbol{r}_i(D_i), \qquad i = 1, 2, \ldots, n. \tag{99}$$

*Define the diameter of the partition as*

$$d(P) := \max\{d(D_1), \ldots., d(D_n)\}, \tag{100}$$

*where for a set $E$, $d(E) := \sup_{\boldsymbol{x}, \boldsymbol{y} \in E} \|\boldsymbol{x} - \boldsymbol{y}\|$.*

9. of Frenet-Serret frame in Differential Geometry.

10. Gesammelte Mathematische Abhandlungen, Vol. 2, p. 309. Berlin, Julius Springer, 1890.

*Then we have*

$$\lim_{d(P)\longrightarrow 0} \sum_{i=1}^{n} A(S_i) = \int_D \|\boldsymbol{r}_u \times \boldsymbol{r}_v\| \,\mathrm{d}(u,v). \tag{101}$$

**Proof.** Since $\boldsymbol{r}_u, \boldsymbol{r}_v$ are continuous on $\bar{D}$, they are uniformly continuous and so is $\boldsymbol{r}_u \times \boldsymbol{r}_v$. Note that this guarantees the integrability of $\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|$ on $D$.

For any $\varepsilon > 0$, take $\delta > 0$ such that

$$\forall \|(u_1,v_1)-(u_2,v_2)\| < \delta, \qquad \|\boldsymbol{r}_u(u_1,v_1)\times\boldsymbol{r}_v(u_1,v_1) - \boldsymbol{r}_u(u_2,v_2)\times\boldsymbol{r}_v(u_2,v_2)\| < \frac{\varepsilon}{\mu(D)}. \tag{102}$$

Thanks to Lemma 12 we have

$$\begin{aligned}\left|\sum_{i=1}^{n} A(S_i) - \int_D \|\boldsymbol{r}_u\times\boldsymbol{r}_v\|\,\mathrm{d}(u,v)\right| &= \left|\sum_{i=1}^{n}\left[\mu(D_i)\,\|\boldsymbol{r}_u\times\boldsymbol{r}_v\|(u_i,v_i) - \int_{D_i}\|\boldsymbol{r}_u\times\boldsymbol{r}_v\|\,\mathrm{d}(u,v)\right]\right| \\ &\leqslant \sum_{i=1}^{n}\int_{D_i} \big|\|\boldsymbol{r}_u\times\boldsymbol{r}_v\|(u_i,v_i) - \|\boldsymbol{r}_u\times\boldsymbol{r}_v\|(u,v)\big|\,\mathrm{d}(u,v) \\ &< \sum_{i=1}^{n}\int_{D_i}\frac{\varepsilon}{\mu(D)}\,\mathrm{d}(u,v) \\ &= \varepsilon\,\frac{\sum_{i=1}^{n}\mu(D_i)}{\mu(D)} = \varepsilon. \end{aligned} \tag{103}$$

Thus ends the proof. □

**Exercise 20.** Prove that if $\boldsymbol{f}, \boldsymbol{g}$ are uniformly continuous on a set $A$, then so are $\boldsymbol{f}\cdot\boldsymbol{g}$, $\boldsymbol{f}\times\boldsymbol{g}$.

**Exercise 21.** Prove that there is $M > 0$ such that

$$\max\{d(S_1), \ldots, d(S_n)\} \leqslant M\, d(P). \tag{104}$$

## 4.4. Independence of parametrization for surface integrals

DEFINITION 46. *Let $D$, $E \subset \mathbb{R}^2$ be non-empty. A $C^1$ map $T$: $E \mapsto D$ is called an admissible transformation if*

*i. it is injective, and*

*ii. The Jacobian is positive for all $(u,v) \in E$ (or all negative, that is does not change sign).*

**Exercise 22.** Show that ii $\not\Longrightarrow$ i.

THEOREM 47. (INDEPENDENCE OF PARAMETRIZATION FOR SURFACE INTEGRALS OF THE FIRST KIND) *Let $\boldsymbol{r}\colon D \mapsto \mathbb{R}^3$ be a $C^1$ surface. Let $\boldsymbol{R}\colon E \mapsto \mathbb{R}^3$ be defined through*

$$\boldsymbol{R}(u,v) := \boldsymbol{r}(T(u,v)). \tag{105}$$

*Then any function $f$ integrable on $\boldsymbol{r}$ is also integrable on $\boldsymbol{R}$. Furthermore*

$$\int f(\boldsymbol{r})\,\|\boldsymbol{r}_u\times\boldsymbol{r}_v\|\,\mathrm{d}(u,v) = \int_E f(\boldsymbol{R})\,\|\boldsymbol{R}_s\times\boldsymbol{R}_t\|\,\mathrm{d}(s,t). \tag{106}$$

**Proof.** Exercise. □

Theorem 48. (Independence of parametrization for surface integrals of the second kind) *Let $S$ be a $C^1$ surface parametrized in two ways $\boldsymbol{r}_1: D_1 \mapsto \mathbb{R}^3$ and $\boldsymbol{r}_2: D_2 \mapsto \mathbb{R}^3$. Further assume that $\boldsymbol{r}_1^{-1} \circ \boldsymbol{r}_2: D_2 \mapsto D_1$ is a $C^1$ bijection with non-vanishing Jacobian. Then we have*

$$\frac{\boldsymbol{r}_{1u} \times \boldsymbol{r}_{1v}}{\|\boldsymbol{r}_{1u} \times \boldsymbol{r}_{1v}\|}(u_1, v_1) = \pm \frac{\boldsymbol{r}_{2u} \times \boldsymbol{r}_{2v}}{\|\boldsymbol{r}_{2u} \times \boldsymbol{r}_{2v}\|}(u_2, v_2) \tag{107}$$

*where $\boldsymbol{r}_1(u_1, v_1) = \boldsymbol{r}_2(u_2, v_2)$, and $+$ is taken if the Jacobian is always positive, and $-$ is taken if the Jacobian is always negative.*

**Proof.** First notice that, since the Jacobian of $\boldsymbol{r}_1^{-1} \circ \boldsymbol{r}_2$ is by assumption continuous, the fact that it is non-vanishing implies that it is either positive everywhere or negative everywhere.

Define

$$\begin{pmatrix} U(u,v) \\ V(u,v) \end{pmatrix} = \left(\boldsymbol{r}_1^{-1} \circ \boldsymbol{r}_2\right)(u,v). \tag{108}$$

Then we have

$$\boldsymbol{r}_1(U(u,v), V(u,v)) = \boldsymbol{r}_2(u,v). \tag{109}$$

Direct calculation gives

$$\boldsymbol{r}_{2u} \times \boldsymbol{r}_{2v} = \det\left(\frac{\partial(U,V)}{\partial(u,v)}\right) \boldsymbol{r}_{1U} \times \boldsymbol{r}_{1V}. \tag{110}$$

The conclusion then follows. □

## 4.5. Surface integrals in $\mathbb{R}^N$

Now we briefly discuss surfaces in $\mathbb{R}^N$ and integration on them. It is clear that we should first understand the surface area of $T(D)$ where $D$ is the unit cube in $\mathbb{R}^{N-1}$ and $T$ is a linear mapping $\mathbb{R}^{N-1} \mapsto \mathbb{R}^N$. We note that a reasonable definition of Area($T(D)$) would be

$$\forall \boldsymbol{x} \in \mathbb{R}^N, \qquad \mathrm{Vol}(C) = A(T(D)) \, |\boldsymbol{n} \cdot \boldsymbol{x}| \tag{111}$$

where $C$ is the parallelogram spanned by $\{T(\boldsymbol{e}_1), \ldots, T(\boldsymbol{e}_{N-1}), \boldsymbol{x}\}$.

Now recall that we know this volume to be

$$\det\left(\, T(\boldsymbol{e}_1) \quad \cdots \quad T(\boldsymbol{e}_{N-1}) \quad \boldsymbol{x} \,\right). \tag{112}$$

**Exercise 23.** Prove that

$$A(T(D)) = \left\| \begin{pmatrix} \det A_1 \\ \vdots \\ \det A_N \end{pmatrix} \right\| \tag{113}$$

where each matrix $A_i$ is obtained from the matrix $\left(\, T(\boldsymbol{e}_1) \ \cdots \ T(\boldsymbol{e}_{N-1}) \,\right)$ by deleting the $i$-th row.

Theorem 49. *Let $S$ be an $N-1$ dimensional surface in $\mathbb{R}^N$ defined through $\boldsymbol{r}: D \subseteq \mathbb{R}^{N-1} \mapsto \mathbb{R}^N$ with $\boldsymbol{r} \in C^1$. Then the surface area of $S$ is given by*

$$A(S) = \int_D \left\| \begin{pmatrix} \det A_1 \\ \vdots \\ \det A_N \end{pmatrix} \right\| \mathrm{d}(u_1, \ldots, u_{N-1}) \tag{114}$$

*where each matrix $A_i$ is obtained from the matrix $\left(\, \boldsymbol{r}_{u_1} \ \cdots \ \boldsymbol{r}_{u_{N-1}} \,\right)$ by deleting the $i$-th row.*

**Remark 50.** Similarly one can define the integration of scalar and vector functions.

This argument can be easily generalized to the situation of a $M$-dimensional surface in $N$-dimensional space.

THEOREM 51. *Let $S$ be an $M$-dimensional surface in $\mathbb{R}^N$ defined through $\boldsymbol{r}: D \subseteq \mathbb{R}^M \mapsto \mathbb{R}^N$ with $\boldsymbol{r} \in C^1$. Then the surface area of $S$ is given by*

$$A(S) = \int_D \sqrt{\det(J^T J)}\, \mathrm{d}(u_1, \ldots, u_M) \tag{115}$$

*where $J := \left( \frac{\partial r_i}{\partial u_j} \right)$ is the Jacobian matrix.*

**Exercise 24.** Show that Theorem 49 is a corollary to the above result.

**Proof.** We will sketch the first step of the proof, that is proving (115) for linear $\boldsymbol{r}$. In this case we have

$$\boldsymbol{r}(\boldsymbol{u}) := J\boldsymbol{u}. \tag{116}$$

Note that $J$ is a $N \times M$ matrix. Now write $J = (\ \boldsymbol{b}_1, \ldots, \boldsymbol{b}_M\ )$. Let $\{\boldsymbol{b}_{M+1}, \ldots, \boldsymbol{b}_N\}$ be an orthonormal set of vector spanning $[\mathrm{span}\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_M\}]^\perp$.

Define

$$\boldsymbol{R}(u_1, \ldots, u_N) := \boldsymbol{r}(u_1, \ldots, u_M) + u_{M+1}\boldsymbol{b}_{M+1} + \cdots + u_N \boldsymbol{b}_N. \tag{117}$$

We see that

$$\mu_M(\boldsymbol{r}(D)) = \mu_N(\boldsymbol{R}(D \times [0,1]^{N-M})) \tag{118}$$

where $\mu_M, \mu_N$ denote Jordan measures in $\mathbb{R}^M$ and $\mathbb{R}^N$ respectively. Thus all we need to do is to calculate

$$\left| \det \frac{\partial(R_1, \ldots, R_N)}{\partial(u_1, \ldots, u_N)} \right|. \tag{119}$$

Simple calculation gives

$$\frac{\partial(R_1, \ldots, R_N)}{\partial(u_1, \ldots, u_N)} = (\ J\ \ B\ ), \qquad B := (\ \boldsymbol{b}_{M+1}\ \ \cdots\ \ \boldsymbol{b}_N\ ). \tag{120}$$

Thus we have

$$\left| \det \frac{\partial(R_1, \ldots, R_N)}{\partial(u_1, \ldots, u_N)} \right| = \sqrt{\det[(\ J\ \ B\ )^T(\ J\ \ B\ )]} = \sqrt{\det \begin{pmatrix} J^T J & 0 \\ 0 & I \end{pmatrix}} = \sqrt{\det(J^T J)}. \tag{121}$$

Thus ends the proof for linear $\boldsymbol{r}$. $\square$

**Exercise 25.** Complete the proof.

**Remark 52.** For surface integrals in $\mathbb{R}^N$, it is more convenient (but not absolutely necessary) to adopt the framework of differential forms. Therefore we do not discuss it here. There are introductory sections on differential forms in many advanced calculus books, e.g. (FOLLAND) §5.9.

# 5. MORE EXERCISES AND PROBLEMS

For (many many) more exercises on calculation of line and surface integrals, see (DEMIDOVICH) ,(EFIMOV) .

## 5.1. Basic exercises

### 5.1.1. Surfaces in $\mathbb{R}^3$

**Exercise 26.** Let $\boldsymbol{r}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$. Prove that

$$\boldsymbol{r}_u \times \boldsymbol{r}_v = \begin{pmatrix} \det \frac{\partial(y,z)}{\partial(u,v)} \\ \det \frac{\partial(z,x)}{\partial(u,v)} \\ \det \frac{\partial(x,y)}{\partial u, v} \end{pmatrix}. \tag{122}$$

Also discuss the condition $\boldsymbol{r}_u \times \boldsymbol{r}_v \neq \boldsymbol{0}$ in the context of inverse function theorem.

**Exercise 27.** Let $S$ be parametrized by $\boldsymbol{r} = \begin{pmatrix} x(\rho,\theta) \\ y(\rho,\theta) \\ z(\rho,\theta) \end{pmatrix}$ where $(\rho,\theta) \in D$ is polar coordinates. Then

$$A(S) = \int_D \left[\rho^2 \left(\frac{\partial z}{\partial \rho}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 + \rho^2\right]^{1/2} \mathrm{d}(\rho,\theta). \tag{123}$$

### 5.1.2. Surface integral of scalar functions (first kind/type)

**Exercise 28.** (DEMIDOVICH) Compute the surface integral

$$\int_S (x+y+z)\,\mathrm{d}S \tag{124}$$

where $S$ is the boundary of the unit cube $\{(x,y,z) | 0 \leqslant x,y,z \leqslant 1\}$. (Ans:[11] )

**Exercise 29.** (DEMIDOVICH 2347) Evaluate

$$\int_S (x^2+y^2)\,\mathrm{d}S \tag{125}$$

where $S$ is the sphere $x^2+y^2+z^2=R^2$. (Hint:[12] )

**Exercise 30.** (DEMIDOVICH 2348) Evaluate

$$\int_S \sqrt{x^2+y^2}\,\mathrm{d}S \tag{126}$$

where $S$ is the lateral surface of the cone $\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{b^2}=0$, $0 \leqslant z \leqslant b$.

### 5.1.3. Surface integral of vector functions (second kind/type)

**Exercise 31.** Let $f(x,y,z)\colon \mathbb{R}^3 \mapsto \mathbb{R}$ be $C^1$. Let $a \in \mathbb{R}$ be such that $Df$ is non-singular at every $(x,y,z)$ satisfying $f(x,y,z)=a$. (Such $a$ are called "regular values" of $f$). Then $S := \{(x,y,z) | f(x,y,z)=a\}$ is orientable. (Hint:[13] )

**Exercise 32.** Calculate

$$I = \int_S x^3\,\mathrm{d}y \wedge \mathrm{d}z + y^3\,\mathrm{d}z \wedge \mathrm{d}x + z^3\,\mathrm{d}x \wedge \mathrm{d}y \tag{127}$$

where $S$ is the ellipsoid

$$\left\{(x,y,z) | \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\right\} \tag{128}$$

11. 9.
12. Symmetry.
13. Consider the gradient of $f$.

oriented by the outer normal. (Ans:[14] )

**Exercise 33.** Calculate

$$\int_S \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \mathbf{d}\boldsymbol{S} \tag{129}$$

where $S = \left\{ (x, y, z) | \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, x, y, z \geqslant 0 \right\}$. (Ans:[15] )

**Exercise 34.** (EFIMOV) Calculate

$$\int_S y \, \mathrm{d}z \wedge \mathrm{d}x \tag{130}$$

where $S$ is the first octant part of $x + y + z = a$. (Ans:[16] )

**Exercise 35.** (EFIMOV) Calculate

$$\int_S \frac{\mathrm{d}x \wedge \mathrm{d}y}{z} \tag{131}$$

where $S$ is the sphere $x^2 + y^2 + z^2 = a^2$ with outer normal. (Hint:[17] )

## 5.2. More exercises

**Exercise 36.** Let $S$ be a surface of revolution and $C$ its generating curve. Let $C$ be parametrized by arc length and let $\rho(s)$ be the distance between the point on $C$ at length $s$ and the rotation axis. Prove Pappus' Theorem

$$A(S) = 2\pi \int_0^l \rho(s) \, \mathrm{d}s. \tag{132}$$

**Exercise 37.** Let $S$ be the unit sphere $\{x^2 + y^2 + z^2 = 1\}$. Prove that

$$\int_S f(a\,x + b\,y + c\,z) \, \mathrm{d}S = 2\pi \int_{-1}^1 f\left(u\sqrt{a^2 + b^2 + c^2}\right) \mathrm{d}u. \tag{133}$$

(Hint:[18] )

## 5.3. Problems

**Problem 1.** Prove that the surface area of $C^1$ surfaces are independent of parametrization.

14. $\frac{4}{5}\pi\,a\,b\,c\,(a^2 + b^2 + c^2)$.
15. $\frac{3}{8}\pi\,a\,b\,c$.
16. $a^3/6$.
17. The outer normal is $(x/a, y/a, z/a)$. Write the integral to integral of the first kind.
18. First reduce to the case $a^2 + b^2 + c^2 = 1$. Then do a change of coordinates.