# Math 317 Week 06: Ordinal Arithmetics

March 18, 2014

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## References


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# 1. ORDERING

## 1.1. Relation

DEFINITION 1. (RELATION) *A "relation" $R$ from a set $X$ to a set $Y$ is a subset $A_R \subseteq X \times Y$. We write $x R y$ (or $R(x, y)$) if and only if $(x, y) \in A_R$.*

*The domain and range of a relation $R$ is defined as the projection of $A_R$ to $X$ and $Y$:*

$$\operatorname{dom} R: = \{x \in X : \exists y \in Y \quad x R y\}; \qquad \operatorname{ran} R := \{y \in Y : \exists x \in X \quad x R y\}. \tag{1}$$

NOTATION 2. *When $X = Y$ we say a "relation" $R$ in $X$.*

**Exercise 1.** Let $x R y$ be $x < y$ and $X = \mathbb{R}$. What is $A_R$? (Hint:[1] )

**Exercise 2.** Let $x R y$ be defined as $x - 3 < y < x^2$. What is $A_R$? (Hint:[2] )

DEFINITION 3. (FUNCTION) *A "function" $f: X \mapsto Y$ is a relation from $X$ to $Y$ such that*

$$\forall x_0 \in X, \qquad A_R \cap \{(x_0, y) |\, y \in Y\} \text{ has exactly one element.} \tag{2}$$

## 1.2. Ordering: partial; total; well

DEFINITION 4. (PARTIAL ORDERING) *A relation $R$ in a set $X$ is said to be a partial ordering (or simply an ordering) of $X$ if and only if it is*

*i. reflexive: $x \in X \Longrightarrow x R x$;*

*ii. anti-symmetric: $(x, y \in X, x \neq y, x R y) \Longrightarrow \neg (y R x)$;*

*iii. transitive: $(x, y, z \in X, x R y, y R z) \Longrightarrow x R z$.*

**Exercise 3.** Prove that $\leqslant$ is a partial ordering on $\mathbb{R}$.

**Exercise 4.** Let $X = \mathbb{R}^2$ and $(x_1, x_2) R (y_1, y_2)$ be defined as

$$x_1 < y_1 \text{ or } (x_1 = y_1 \text{ and } x_2 < y_2) \text{ or } (x_1 = y_1, x_2 = y_2). \tag{3}$$

Prove that it is an ordering.

**Exercise 5.** Prove that $(x, y) \leqslant (a, b)$ defined as $x \leqslant a$, $y \leqslant b$ is a partial ordering of $\mathbb{R}^2$.

**Remark 5.** In the following we will use $\leqslant$ to denote partial ordering.

DEFINITION 6. *Let "$\leqslant$" be a partial ordering in a set $X$. We define $x < y$, $x \geqslant y$, $x > y$ as*

$$x \leqslant y, x \neq y; \qquad y \leqslant x; \qquad y < x \tag{4}$$

*respectively.*

DEFINITION 7. *Let $(X, \leqslant)$ be a partially ordered set. $x_0 \in Y \subseteq X$ is said to be*

- *a **least element** of $Y$ if and only if for every $y \in Y$, $x_0 \leqslant y$.*
- *a **minimal element** of $Y$ if and only if there is no $y \in Y$ such that $y < x_0$.*

**Remark 8.** "Least" means smaller than others; "minimal" means no other element is smaller.

**Exercise 6.** Define "maximal" and "greatest" elements. (Hint:[3] )

1. $(x, y), x < y$.
2. $(x, y),\ x - 3 < y < x^2$.
3. greatest $\longleftrightarrow$ least; maximal $\longleftrightarrow$ minimal.

**Example 9.** Let $\mathbb{R}^2$ be equipped with the ordering $(x,\ y) \leqslant (a,\ b) \Longleftrightarrow x \leqslant a,\ y \leqslant b$. Let $Y = \{x \geqslant 0, y \geqslant 0, x^2 + y^2 \geqslant 1\}$. Then

- Least element: No least element;
- Minimal element: $\{x \geqslant 0, y \geqslant 0, x^2 + y^2 = 1\}$.

**Exercise 7.** Find the least elements and minimal elements for $Z := \{x^2 + y^2 \leqslant 1\}$. (Ans:[4] )

**Exercise 8.** Prove that

a) Any least element must also be minimal; But a minimal element may not be a least element.

b) The least element, if it exists, is unique. (Hint:[5] )

**Example 10.** Let $X$ be a set. Then $\subseteq$ is a partial ordering on the power set $\mathcal{P}(X)$.

DEFINITION 11. (TOTAL ORDERING) *A relation $\leqslant$ on a set $X$ is said to be a total ordering (or simple ordering) if and only if*

*i. it is a partial ordering, and*

*ii. it is strongly connected, that is*

$$\forall x, y \in X, \qquad \text{either } x \leqslant y \text{ or } y \leqslant x. \tag{5}$$

**Exercise 9.** Let $(X, \leqslant)$ be a totally ordered set. Prove that any minimal element is also a least element. (Hint:[6] )

**Remark 12.** A totally ordered set is also called a "chain".

DEFINITION 13. (WELL-ORDERING) *A relation $\leqslant$ on a set $X$ is said to be a well-ordering of $X$ if and only if*

*i. it is a total ordering, and*

*ii. every non-empty subset of $X$ has a least element.*

**Exercise 10.** Consider $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{R}^2$ with the lexicographical ordering. Which are partially ordered? totally ordered? well-ordered? (Ans:[7] )

PROPOSITION 14. *A linearly ordered set is well-ordered if and only if there is no infinite decreasing sequence.*

**Proof.** Exercise. (Hint:[8] ) $\square$

## 1.3. Similarity

DEFINITION 15. (SIMILARITY) *Two ordered sets are similar (ordinally isomorphic) if and only if there is a bijection $f: X \mapsto Y$ such that the order is preserved. If $X$ and $Y$ are similar we say they belong to the same "order-type".*

THEOREM 16. *If $(X, \leqslant)$ is a partially ordered set, then there is a set $Y \subseteq 2^X$ such that $(X, \leqslant)$ and $(Y, \subseteq)$ are similar.*

**Proof.** Set

$$Y := \{X_a |\, a \in X\} \tag{6}$$

4. No least; Minimal: $x^2 + y^2 = 1$, $x \leqslant 0$ or $y \leqslant 0$.
5. Assume the contrary, there are $x_1 \neq x_2$ both least. Consider the relation between $x_1, x_2$.
6. If $y < x$ is not true, then $x \leqslant y$.
7. Partial: $\mathbb{R}^2$; total: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$; well: $\mathbb{N}$.
8. If $a_1 > a_2 > a_3 > \cdots$ is infinite, then the subset $\{a_1, a_2, a_3, \ldots\}$ does not have least element; For the other direction let $A \subseteq X$ be nonempty. Take any $a_1 \in A$, if it is least, done; Otherwise there must be $a_2 < a_1$ ...

where

$$X_a := \{x \in X \mid x \leqslant a\}. \tag{7}$$

Then we define

$$f: X \mapsto Y \qquad a \mapsto X_a. \tag{8}$$

The following is left as exercise. □

**Exercise 11.** Finish the proof of the theorem.

**Example 17.** Let $A = \mathbb{N}$ with the natural order; Let $B = \mathbb{N}$ with the following order:

$$2 <_B 1 <_B 4 <_B 3 <_B 6 <_B 5 <_B \cdots \tag{9}$$

Then $A, B$ are similar.

Now let $C = \mathbb{N}$ with the following order:

$$1 <_C 3 <_C 5 <_C \cdots <_C 2 <_C 4 <_C 6 <_C \cdots \tag{10}$$

Then $A, C$ are not similar. To justify this, we assume the contrary: Let $f: A \mapsto C$ be a bijection preserving order. Then there is $n_0 \in \mathbb{N}$ such that $f(n_0) = 2$. Now consider $f(n_0 - 1)$. Since $f$ preserves order, $f(n_0 - 1)$ must be an odd number, denote it by $2\,k_0 - 1$. Finally let $f(m_0) = 2\,k_0 + 1$. Now we have:

$$f(n_0 - 1) = 2\,k_0 - 1 <_C 2\,k_0 + 1 = f(m_0) <_C 2 = f(n_0) \Longrightarrow n_0 - 1 < m_0 < n_0 \tag{11}$$

which is not possible. Contradiction.

**Exercise 12.** Prove that $\mathbb{N}$ and $\mathbb{Q}$ do not have the same order type. (Hint:[9] )

**Exercise 13.** Do $\mathbb{Z}$ and $\mathbb{Q}$ have the same order type? Justify your answer. (Hint:[10] )

THEOREM 18. *Let $A, B$ be two ordered sets. If $|A| = |B|$, then there is a re-ordering making $A$ of the same order-type as $B$.*

**Proof.** Exercise. (Hint:[11] ) □

**Example 19.** Re-order $\mathbb{Z}$ so that it has the same order-type as $\mathbb{N}$.

We introduce the following ordering for $m, n \in \mathbb{Z}$:

$$m \prec n \text{ if and only if } |m| > |n| \text{ or } |m| = |n| \text{ and } m > 0, n < 0. \tag{12}$$

NOTATION. *Sometimes in the definition of the new ordering, the usual ordering of numbers is involved. In such situation to avoid confusion, sometimes a different symbol, such as $\preccurlyeq$, $\npreceq$, $\trianglelefteq$, or $\blacktriangleleft$, are used to denote the new ordering.*

***Re-order***

Note that when re-ordering a set, its elements must all be involved. For example, $2 < 4 < 3 < 5$ is not a re-ordering of $1 < 2 < 3 < 4 < 5$.

**Exercise 14.** Re-order $\mathbb{N}$ so that it has the same order-type as $\mathbb{Z}$. (Hint:[12] )

**Exercise 15.** Re-order $(0, 1]$ so that it has the same order-type as $[0, 100)$ equipped with the natural order. (Hint:[13] )

9. Assume otherwise, then there is $f: \mathbb{N} \mapsto \mathbb{Q}$ order-preserving. Consider $f(1)$.
10. Assume they are. There is $r \in \mathbb{Q}$ between $f(0), f(1)$.
11. Let $f: A \mapsto B$ be a bijection. Then order $B$ as follows: $b_1 \leqslant b_2$ is defined by $f^{-1}(b_1) \leqslant f^{-1}(b_2)$.
12. $\cdots < 4 < 2 < 1 < 3 < 5 < \cdots$.
13. $x, y \in (0, 1]$, $x \prec y \Longleftrightarrow x > y$.

# 2. SEGMENTAL SETS

## 2.1. The construction of $\mathbb{N}$

von Neumann suggested the following construction of the set of natural numbers starting from the empty set $\varnothing$.[14]

DEFINITION 20. *Define iteratively:*

$$0 := \varnothing; \qquad 1 := \{0\}; \qquad 2 := \{0, 1\}; \qquad 3 := \{0, 1, 2\}; \ldots \tag{13}$$

*In general,*

$$n + 1 := \{1, 2, \ldots, n\}. \tag{14}$$

**Exercise 16.** Prove that $\mathbb{N}$ as defined above is well-ordered. (Hint:[15] )

We see that in fact

$$1 = \{\varnothing\}; \qquad 2 = \{\varnothing, \{\varnothing\}\}; \qquad 3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} \tag{15}$$

and so on.

THEOREM 21. $\mathbb{N}$*, as constructed above, satisfies the following:*

$$m, n \in \mathbb{N}, \qquad m < n \Longleftrightarrow m \in n. \tag{16}$$

**Proof.** Exercise. □

**Problem 1.** Verify that the $\mathbb{N}$ defined above satisfies Peano's five axioms.

## 2.2. Segments

DEFINITION 22. *Let $(X, \leqslant)$ be well-ordered. Then for any $a \in X$, the subset*

$$X_a := \{x \in X \,|\, x < a\} \tag{17}$$

*is called a "segment" of $X$.*

THEOREM 23. *Let $(X, \leqslant)$ be well-ordered. Let $X_a$ be a segement of $X$. Then $X$ and $X_a$ are not similar.*

**Proof.** Assume the contrary, that is there is a similarity mapping (ordinal isomorphism) $f: X \mapsto X_a$. Now set $x_0 := a$, and $x_1 := f(a) \in X_a$. Clearly $x_1 < x_0$. Now define

$$x_2 := f(x_1), \ldots, x_n := f(x_{n-1}), \ldots \tag{18}$$

we have $x_0 > x_1 > x_2 > \cdots$ which form a non-empty subset of $X$ without a least element, thus contradicting the hypothesis that $(X, \leqslant)$ is well-ordered. □

14. It turns out that, if there is a set, then there must be empty set. To see this, let $A$ be a set. Now we define $B := \{x \in A: x \neq x\}$. Clearly $B$ has no element and is therefore the empty set.

15. Take intersection.

## 2.3. Segmental sets

DEFINITION 24. (SEGMENTAL SET) *A well-ordered set $(X, \leqslant)$ is said to be "segmental" if and only if*

$$\forall a \in X, \qquad a = X_a := \{b \in X \mid b < a\}. \tag{19}$$

**Example 25.** $\mathbb{N}$ is segmental.

***Everything are sets!***

On first encounter (19) may be hard to understand or may even look nonsensical – how can an element $a$ be equal to a subset $X_a$? The reason for this difficulty in understanding is that we usually think of elements of a set as "atoms", which can aggregate to form subsets and so on. However, in the set-theoretic approach to mathematics, there is nothing but sets, sets of sets, sets of sets of sets, .... Therefore relations like (19) makes perfect sense.

**Remark 26.** Note that in this definition, the order relation is already "encoded" in the elements themselves – every element "carries" all the elements that smaller than it. Therefore for segmental sets, there is no need to specify the order relation $\leqslant$ anymore.

**Exercise 17.** Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be segmental. Let $a, b \in X \cap Y$ (set intersection, ignoring the order relation). Then

$$a <_X b \Longleftrightarrow a <_Y b. \tag{20}$$

(Hint:[16] )

Thus from now on we will omit $\leqslant$ when talking about segmental sets.

THEOREM 27. *Let $X$ be segmental. Let $Y \subset X$ be a proper subset (that is $Y \subsetneq X$). Then $Y$ is a segment of $X$ if and only if $Y$ is segmental.*

**Proof.**

- If.

  Since $Y$ is a proper subset of $X$, $Z := X - Y$ is non-empty. Because $X$ is well-ordered, $Z$ has a least element $a$. We now prove that $Y = X_a$.

  - $Y \subseteq X_a$. Take any $b \in Y$. Note that as both $X, Y$ are segmental, we have

    $$X_b = b = Y_b. \tag{21}$$

    Since $a \in X - Y$ clearly $b \neq a$. If $b > a$, then we have $a \in X_b = Y_b$ which by definition means $a \in Y$. Contradiction. Therefore $b < a \Longrightarrow b \in X_a$.

  - $X_a \subseteq Y$. Since $a$ is the least element in $Z$, we have, for any $x \in X$,

    $$x \in Z \Longrightarrow x \geqslant a. \tag{22}$$

    Therefore

    $$X_a \cap Z = \varnothing \Longrightarrow X_a \subseteq Y. \tag{23}$$

- Only if.

16. $a <_X b$ by definition is $a \in b$. $a <_Y b$ by definition is also $a \in b$. This result may be easier to understand if we realize the later result that there is exactly one increasing sequence of segmental sets, starting from $\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \ldots$.

Let $Y = X_a$. Take any $b \in Y$. Then $b < a$ and thus

$$Y_b := \{y \in Y \mid y < b\} = \{x \in X \mid x < a, x < b\} = \{x \in X \mid x < b\} = X_b = b. \tag{24}$$

Thus ends the proof. □

Theorem 28. *Let $X$ and $Y$ be segmental sets. Then $X \cap Y$ is also segmental.*

**Proof.** For any $a \in X \cap Y$, we have $X_a = a = Y_a \Longrightarrow X_a = Y_a \subseteq X \cap Y \Longrightarrow a = (X \cap Y)_a$. □

Theorem 29. *Let $X, Y$ be segmental sets. Then exactly one of the following three holds:*

$$X = Y; \qquad \exists b \in Y, X = Y_b; \qquad \exists a \in X, Y = X_a. \tag{25}$$

**Proof.** We assume $X \neq Y$, and try to prove that either $X \cap Y = Y$ or $X \cap Y = X$.

Assume the contrary, that is $X \cap Y \subsetneq Y$ and $X \cap Y \subsetneq X$. Then $X_a = X \cap Y = Y_b$ for some $a \in X$, $b \in Y$. But since $X, Y$ are segmental, we have

$$a = X_a = Y_b = b \Longrightarrow a = b \in X \cap Y. \tag{26}$$

Contradiction! □

**Remark 30.** In particular, we have, for any infinite segmental set $X$, $\mathbb{N} \subseteq X$.

**Exercise 18.** Let $X$ be any non-empty segmental set. Prove that $\varnothing \in X$. (Hint:[17] )

**Exercise 19.** Let $X, Y$ be segmental. Assume that they are similar. Then $X = Y$. (Hint:[18] )

Theorem 31. *Every well-ordered set is similar to one and only one segmental set.*

**Proof.** The "only one" part follows immediately from Exercise 19. Now we show that if $(X, \leqslant)$ is a well-ordered set, then $X$ is similar to some segmental set $Y$. This is done through construction of the similarity mapping $f: X \mapsto Y$ using transfinite induction[19] in two steps.

i. Since $(X, \leqslant)$ is well-ordered, there is a least element $a$. We define

$$f(a) = \varnothing. \tag{27}$$

ii. Now assume that $f$ has been defined for a segment $X_a$, then $f(X_a)$ is also a segment of $Y$ which means $f(X_a) = Y_b$ for some $b$. We define now

$$f(a) = b. \tag{28}$$

By transfinite induction $f$ is defined on the whole $X$. The remaining part of the proof is left as exercise. □

**Exercise 20.** Finish the proof for the theorem.

17. Segmental sets are well-ordered. Consider the least element $a = \{b \in X, b < a\}$.

18. Otherwise we have a segmental set similar to its own segment.

19. See §4.2.

# 3. ORDINAL NUMBERS AND ORDINAL ARITHMETICS

## 3.1. Ordinal numbers

DEFINITION 32. (ORDINAL NUMBER) *Let $X$ be any well-ordered set. The unique segmental set similar to $X$ is called the ordinal number of $X$, and denoted* $\mathrm{Ord}(X)$.

It is clear that

- Two well-ordered sets are similar if and only if they have the same ordinal number;
- If two well-ordered sets are not similar, then one is similar to a segment of the other, or equivalently the ordinal number of one set is strictly smaller than the ordinal number of the other.

NOTATION 33. *We will use lower case Greek letters to denote ordinal numbers.*

**Example 34.** $\mathbb{N}$ is a segmental set and is therefore an ordinal number. We denote it by $\omega$. Then the set

$$\{1,2,3,\ldots,\omega\} \tag{29}$$

is also segmental and we denote it by $\omega+1$. And so on.

Thus we count

$$1,2,3,\ldots,n,\ldots,\omega,\omega+1,\ldots \tag{30}$$

DEFINITION 35. (COMPARISON OF ORDINAL NUMBERS) *Let $\alpha$, $\beta$ be ordinal numbers. We define $\alpha<\beta$ by $\alpha\subset\beta$, and $\alpha>\beta$ by $\beta<\alpha$.*

**Exercise 21.** $\alpha<\beta$ if and only if $\alpha\in\beta$.

THEOREM 36. *Consider the class $W$ of all ordinal numbers. It is segmental.*

**Proof.** We first prove $W$ is well-ordered. Let $U$ be any non-empty subset of $W$. If $U$ does not have a least element, we could find $x_0\supset x_1\supset x_2\supset\cdots$. But since each $x_i$ is segmental, we have

$$x_0\ni x_1\ni x_2\ni\cdots \tag{31}$$

This contradicts the axiom of foundation in the ZF Axiom system.

Now we prove $W$ is segmental. Let $x\in W$, we try to prove $x=W_x:=\{y\in W, y\subset x\}$.

- $x\subseteq W_x$. Take any $z\in x$. Since $x$ is segmental, $z=x_z$ is a segmental proper subset of $x$. Thus $z$ is also an ordinal number and therefore $z\in W$. Therefore $z\in W_x$.
- $W_x\subseteq x$. Take any $z\in W_x$. By definition we have $z\subset x$ and $z$ is a segmental set. By Theorem 29 $z=x_z\in x$. □

**Remark 37.** Note that although all segments of $W$ are sets, $W$ itself is not a set. Otherwise it would be an ordinal number. Denote this number by $\lambda$. We have $W$ is similar to its own segment $W_\lambda$ which contradicts Theorem 23.

Now we can count and index any well-ordered set: Let $(A, \leqslant)$ be well-ordered. Then there is a unique ordinal number $\alpha$ for it. Then we can index the elements in $A$ as

$$A = \{a_\xi | \xi \in W_\alpha\}. \tag{32}$$

## 3.2. Addition

Note that for every ordinal number $\alpha$, we can define its "successor" $\alpha'$ (or $\alpha + 1$) as the least element of $B := \{\beta \in W | \beta > \alpha\}$. In fact, $\alpha' = \alpha \cup \{\alpha\}$. From this clearly one can define addition and then multiplication (just like what we did last semester for $\mathbb{N}$) through transfinite induction[20]. However, it is more practical to use the following definition.

DEFINITION 38. *Let $A, B$ be two disjoint totally ordered sets. The "ordered union" $\langle A \cup B \rangle$ is the set $A \cup B$ with the order*

$$x < y \Longleftrightarrow [x, y \in A, x <_A y; \text{ or } x, y \in B, x <_B y \text{ or } x \in A, y \in B]. \tag{33}$$

*The generalization to more than two sets is straightforward.*

**Exercise 22.** Prove that $\langle \mathbb{N}^- \cup \mathbb{N} \rangle$ is similar to $\mathbb{Z}$. Here $\mathbb{N}^{-1} := \{\cdots, -3, -2, -1\}$.

**Exercise 23.** Prove that, if $A, B$ are well-ordered, then so is $\langle A \cup B \rangle$.

DEFINITION 39. *Let $\alpha = \mathrm{Ord}(A)$, $\beta = \mathrm{Ord}(B)$ and $A, B$ disjoint. Then*

$$\alpha + \beta := \mathrm{Ord}(\langle A \cup B \rangle). \tag{34}$$

**Exercise 24.** Prove that $\alpha + \beta$ is well-defined. That is it does not depend on the choices of $A$, $B$ (as long as $A \cap B = \varnothing$).

**Example 40.** We have

$$1 + \omega = \omega \neq \omega + 1. \tag{35}$$

**Exercise 25.** Let $n \in \mathbb{N}$. Prove that $n + \omega = \omega$. Do we have $\omega + 2 = (\omega + 1) + 1$? Justify your answer.

THEOREM 41. *Addition of ordinal numbers is associative:*

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma). \tag{36}$$

**Proof.** Exercise. □

Now our counting ability is extended:

$$0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega, \omega + \omega + 1, \ldots \tag{37}$$

## 3.3. Multiplication

DEFINITION 42. *Let $\alpha, \beta$ be ordinal numbers. We define*

$$\alpha \cdot \beta := \sum_{\xi < \beta} \alpha_\xi \tag{38}$$

20. §4.2.

*where $\alpha_\xi = \alpha$ for all $\xi < \beta$.*

**Example 43.** Let $\alpha = \mathrm{Ord}[(0,1)]$. Then for any $n \in \mathbb{N}$, we have

$$(1+\alpha)\cdot n = 1+\alpha. \tag{39}$$

Furthermore if $\omega := [\mathbb{N}]$, we have

$$(1+\alpha)\cdot\omega = 1+\alpha \tag{40}$$

**Exercise 26.** Prove the following.

a) $1\cdot\omega = \omega = \omega\cdot 1$;

b) $2\cdot\omega = \omega$; $\omega\cdot 2 = \omega+\omega$;

c) $(\omega+1)\cdot\omega = \omega\cdot\omega$; $\omega\cdot(\omega+1) = \omega\cdot\omega+\omega$.

**Exercise 27.** Prove

$$\alpha\cdot(\beta+\gamma) = \alpha\cdot\beta+\alpha\cdot\gamma; \tag{41}$$

Show that it is possible that

$$(\alpha+\beta)\cdot\gamma \neq \alpha\cdot\gamma+\beta\cdot\gamma. \tag{42}$$

THEOREM 44. *Multiplication is associative:*

$$(\alpha\cdot\beta)\cdot\gamma = \alpha\cdot(\beta\cdot\gamma). \tag{43}$$

## 3.4. Power

DEFINITION 45. *The power $\alpha^\beta$ is defined successively as follows:*

- *If $\alpha = 0$, then $\alpha^0 = 1$, $\alpha^\beta = 0$ for all other $\beta$;*
- *If $\alpha > 0$, then $\alpha^0 = 1, \alpha^{\xi+1} = \alpha^\xi\cdot\alpha$, $\alpha^\beta = \lim_{\xi<\beta}\alpha^\xi$ if $\xi$ does not have a predecessor.*

**Example 46.** We have

$$\omega^2 = \omega\cdot\omega; \quad \omega^3 = \omega\cdot\omega\cdot\omega; \qquad \omega^\omega = \lim_{n<\omega}\omega^n. \tag{44}$$

We also have

$$2^\omega = \lim_{n<\omega} 2^n = \omega. \tag{45}$$

**Exercise 28.** Let $n \in \mathbb{N}$. Prove that $n^\omega = \omega$.

THEOREM 47. *We have*

$$\alpha^\beta\cdot\alpha^\gamma = \alpha^{\beta+\gamma}; \qquad (\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}. \tag{46}$$

**Exercise 29.** Show that $\alpha^\gamma\cdot\beta^\gamma = (\alpha\,\beta)^\gamma$ may not hold.

**Example 48.** We have

$$2^\omega\cdot\omega^\omega = \omega\cdot\omega^\omega = \omega^{1+\omega} = \omega^\omega. \tag{47}$$

# 4. Advanced Topics, Notes, and Comments

## 4.1. A brief history of set theory

- Dauben, Joseph W., *The development of Cantorian set theory*, in *From the Calculus to Set Theory, 1630 - 1910: An Introductory History*, edited by I. Grattan-Guinness, pp. 181 – 219.
- Lavine, Shaughan, *Understanding the Infinite*, Harvard University Press, 1994.

Set theory grew out of Georg Cantor's effort to understand the real numbers, Richard Dedekind and Peano's effort to reduce Analysis to Arithmetics, and Gottlob Frege's effort to reduce Arithmetics to Logic.

### 4.1.1. Cantor

Georg Cantor did his Dissertation and Habilitationsschrift, in 1867 and 1869 respectively, under Kummer and Kronecker, in number theory. In 1869 he became Privatdozent in the University of Halle and met Eduard Heine, who was studying the following problem:

> Given two trigonometric series
>
> $$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos(n\,x)+b_n\sin(n\,x)\right]; \qquad \frac{A_0}{2}+\sum_{n=1}^{\infty}\left[A_n\cos(n\,x)+B_n\sin(n\,x)\right], \tag{48}$$
>
> assume that they converge to the same function $f(x)$ on $[-\pi,\pi]$. Does it follow that $a_n=A_n, b_n=B_n$ for all $n$?

Fourier proved that this must be the case. But later Heine realized in 1870 that Fourier had assumed uniform convergence and proceed to prove that the conclusion still holds when uniform convergence occurs only "almost everywhere". Heine suggested this problem to Cantor.

In 1870 Cantor relaxed the condition to everywhere convergence.

In 1871 Then Cantor moved on to prove that the conclusion still holds when the two series differ at finitely many points.

Later in 1872 Cantor realized that infinitely many exceptional points could be allowed. More specifically, he defined for any set $S$, its "derived set"

$$S' := \{\text{All limit points of } S\}. \tag{49}$$

Then he defined higher order derived sets:

$$S'' := (S')', \qquad , S^{(n+1)} := \left(S^{(n)}\right)', \ldots \tag{50}$$

**Exercise 30.** Cantor made the following definitions in 1879. Prove that they are equivalent to the definitions used today.

- A set $P$ is dense in $(a,b)$: $(a,b)\subseteq P'$;
- A set $P$ is isolated: $P\cap P'=\varnothing$;
- A set $P$ is closed: $P\cap P'=P'$.

With these new definitions, he proved:

THEOREM. *Let $E$ be the set of points where the two series differ. Assume that there is $n\in\mathbb{N}$ such that $E^{(n)}$ is finite, then $a_n=A_n, b_n=B_n$ for all $n$.*

**Exercise 31.** Find a set $E\subseteq\mathbb{R}$ such that $E, E^{(1)}$ are infinite, but $E^{(2)}$ is finite.

Now Cantor naturally wondered, what if we define

$$S^{(\infty)} := \cap S^{(n)}, \tag{51}$$

and then

$$S^{(\infty+1)} := \left(S^{(\infty)}\right)' \tag{52}$$

and so on?

**Exercise 32.** Let $S$ be closed. Prove that

$$S - S^{(\infty)} := \{x \in S | \, x \notin S^{(\infty)}\} \tag{53}$$

is countable.

By then his interest shifted from Fourier series to properties of sets of real numbers, and the characteriztion of the continuum $\mathbb{R}$ itself.

**Exercise 33.** If we replace trigonometric series by power series, what kind of result can you prove? Can you explain why this "uniqueness of power series" problem does not seem to lead to any major mathematical discovery?

**Remark 49.** The uniqueness problem for Trigonometric series was more or less completely settled by Henri Lebesgue through his generalized integration theory and the core of this new theory: Theorem of Dominated Convergence.

Cantor first pondered over the problem of comparing sizes of sets, in particulr $\mathbb{N}$ and $\mathbb{R}$. In 1873 he wrote to Dedekind asking for help, while Dedekind replied that he saw no reason why $\mathbb{N}$ and $\mathbb{R}$ should not have the same cardinality. But very soon Cantor proved that $\mathbb{N} < \mathbb{R}$ using the least upper bound property of $\mathbb{R}$.

Next Cantor tried to understand the relation between $\mathbb{R}^N$ and $\mathbb{R}$. A problem that shocked his friends as ridiculous. He finally proved in 1877 that $\mathbb{R}^N \sim \mathbb{R}$.

Cantor then tried very hard to prove the Continuum Hypothesis but only with limited success. His idea was to understand the continuum through perfect sets. At first he thought that any perfect set would contain an interval and therefore has cardinality $\mathfrak{c}$.

**Exercise 34.** Let $S \subseteq \mathbb{R}$. Prove that if $S$ contains an open interval, then $S \sim \mathbb{R}$.

**Exercise 35.** Prove that the Cantor set is a perfect set that does not contain any open interval.

Later Cantor managed to prove that, for any closed set $S$, there is a countable ordinal $\alpha$ such that

$$S^{(\alpha)} = S^{(\alpha+1)} \tag{54}$$

which means $S^{(\alpha)}$ is perfect. He then proved that any perfect set is either empty or having cardinality $\mathfrak{c}$. Thus he has proved:

Every closed set in $\mathbb{R}$ has cardinality $\mathfrak{c}$.

**Exercise 36.** Why proving $S^{(\alpha)} \sim \mathbb{R}$ is enough?

In 1885 Cantor claimed that pure mathematics is nothing other than pure set theory. Later in 1895-97 he finally established a complete theory of cardinal and ordinal arithmetics. In particular he was finally able to related $\mathfrak{c}$ to the $\aleph$'s:

$$\mathfrak{c} = 2^{\aleph_0}. \tag{55}$$

### 4.1.2. Zermelo-Fraenkel Axioms

Ernst Zermelo, as a logician, could not bear with Cantor's lack of rigor in the definitions and tried to put everything on a solid foundation. In light of the various paradoxes, Zermelo proposed a Axiomatic approach which basically tries to generate all the sets that mathematics will ever need from a list of Axioms.

1. Axiom of extensionality.

$$\forall x\,(x \in a \Longleftrightarrow x \in b) \Longrightarrow a = b. \tag{56}$$

2. Axiom of elementary sets.
   - There is a set with no element. Denote by $\varnothing$;
   - For any set $a$, there is a set $\{a\}$ whose only element is $a$;
   - For any sets $a, b$, there is a set $\{a, b\}$ whose only elements are $a, b$.
3. Axiom of selection of subsets (Aussonderung). For any set $a$ and any propositional function (meeting some mild conditions), there is a set $b$ such that
$$x \in b \Longleftrightarrow (x \in a \text{ and } \Phi(x) \text{ is true}). \tag{57}$$
4. Axiom of the power set. For any set $a$ there is a set $\mathcal{P}(A)$ such that
$$x \in \mathcal{P}(A) \Longleftrightarrow x \subseteq a. \tag{58}$$
5. Axiom of the union. For any set $a$, there is a set $\cup a$ such that
$$x \in \cup a \Longleftrightarrow \exists y \in a, x \in y. \tag{59}$$
6. Axiom of choice. If $a$ is a set, and the elements of $a$ are all non-empty and no two of them have any element in common, then there is a set $c$ satisfying
$$\forall x \in a, \qquad c \cap x \text{ has exactly one element.} \tag{60}$$
7. Axiom of infinity. There is a set $a$ such that $\varnothing \in a$, and such that
$$\forall e \in a, \qquad \{e\} \in a. \tag{61}$$
8. Axiom of replacement. If $a$ is a given set, and $\Psi(x, y)$ is a propositional function satisfying for any $x \in a$, there is a unique $y$ such that $\Psi(x, y)$. Then there is a set $a'$ consisting of exactly these $y$'s.
9. Axiom of foundation (Fundierung). Let $a$ be any non-empty set. Then there is $e \in a$ such that $e \cap a = \varnothing$.

**Exercise 37.** Prove that the Axiom of foundation excludes the following situation:
$$x_0 \ni x_1 \ni x_2 \ni x_3 \ni \cdots \ni x_n \ni \cdots. \tag{62}$$

## 4.2. Transfinite induction

THEOREM 50. (TRANSFINITE INDUCTION) *Let $(X, <)$ be a well-ordered set. Let $E \subseteq X$ such that*

*i. The smallest element of $X$ is a member of $E$;*

*ii. For any $x \in X$, if $\forall y < x, y \in E$, then $x \in E$.*

*Then $E = X$.*

**Proof.** Assume the contrary. Denote $F := X - E := \{x \in X | x \notin E\}$. Then $F$ is non-empty and there is $f \in F$ such that
$$\forall y \in F, \qquad y \geqslant f. \tag{63}$$
Now since $f \notin E$, it is not the smallest element in $X$. Thus the set $\{z \in X | z < f\}$ is non-empty. By (63) any such $z \in E$. Consequently $f \in E$. Contradiction. $\square$

**Example 51.** Let $(X,<)$ be a well-ordered set. Let $Y \subseteq X$. Assume there is an order-preserving bijection $f: X \mapsto Y$. Then $x \leqslant f(x)$.

**Proof.** Let $E := \{x \in X | x \leqslant f(x)\} \subseteq X$. We prove that $E = X$ by transfinite induction. First this obviously holds for the smallest element of $X$. Now let $x \in X$. Assume that

$$\forall y < x, \qquad y \in E \Longrightarrow y \leqslant f(y). \tag{64}$$

As $f$ preserves order, we have

$$\forall y < x, \qquad y \leqslant f(y) < f(x). \tag{65}$$

Therefore $x \leqslant f(x)$. The conclusion now follows. □

**Example 52.** Let $(X,<)$ and $(Y,\prec)$ be two well-ordered sets. Then the order-preserving bijection from $X \mapsto Y$, if it exists, is unique.

**Proof.** Let $f, g$ be two such bijections. We prove $\forall x \in X, f(x) = g(x)$. Set $E := \{x \in X | f(x) = g(x)\}$.

First clearly the smallest element of $X$ belongs to $E$. Now let $x \in X$. Assume that

$$\forall y < x, \quad y \in E \text{ that is } f(y) = g(y). \tag{66}$$

Thus we have

$$f(x) > \tag{67}$$

□

**Remark 53.** Note that if $X, Y$ are not well-ordered the conclusion does not hold. For example let $X = Y = \mathbb{Z}$ and $f(x) = x + 1, g(x) = x + 2$.

We can also formulate transfinite induction using ordinal numbers.

> If $P(\alpha)$ is a form of statement that involves an unspecified ordinal number $\alpha$, and if $P(\alpha)$ is true whenever $P(\beta)$ is true for all $\beta < \alpha$, then $P(\alpha)$ is true for every $\alpha$.

**Exercise 38.** Where does the "base" step go?

## 4.3. More on Axiom of Choice

### 4.3.1. An example

First let's see how it is formally used.

THEOREM 54. *Let $X$ be an infinite set. Then there is $Y \subseteq X$ such that $Y \sim \mathbb{N}$.*

**Remark 55.** Note that this cannot be proved by induction.

**Proof.** Let $f: \mathcal{P}(X) - \varnothing \mapsto A$ be a choice function. Let $\mathcal{C}$ be the collection of all finite subsets of $X$. For any $A \in \mathcal{C}$, since $X$ is infinite, $X - A$ is non-empty and therefore we can define a function $g: \mathcal{C} \mapsto \mathcal{C}$ by $g(A) := A \cup \{f(X - A)\}$.

Now we apply transfinite induction to define a function $U: \omega \mapsto \mathcal{C}$:

$$U(0) = \varnothing, \cdots, U(n^+) = U(n) \cup \{f(X - U(n))\}. \tag{68}$$

Then we set $v(n) := f(X - U(n))$. Here $n^+$ is the successor of $n$.

We finally prove that $v$ is one-to-one. Notice that

- $v(n) \notin U(n)$;
- $v(n) \in U(n^+)$.

From this it is clear that $m > n \Longrightarrow v(n) \in U(m)$ but $m \notin U(n)$. Consequently $v(n) \neq v(m)$ whenever $m \neq n$. Thus ends the proof. $\square$

### 4.3.2. Zorn's Lemma

DEFINITION 56. (UPPER BOUND) *Let $(X, \leqslant)$ be partially ordered. Let $Y \subseteq X$. Say $x \in X$ is an upper bound of $Y$ if and only if*

$$\forall y \in Y, \qquad y \leqslant x. \tag{69}$$

DEFINITION 57. (CHAIN) *Let $(X, \leqslant)$ be partially ordered. A subset $Y \subseteq X$ is called a "chain" if and only if $(Y, \leqslant)$ is totally ordered.*

We have the following

> (ZORN'S LEMMA) Let $(X, \leqslant)$ be partially ordered. If every chain in $X$ has an upper bound, then $X$ has a maximal element.

THEOREM 58. *Assuming the Axioms of the Zermelo-Fraenkel set theory, then Zorn's Lemma is equivalent to Axiom of Choice.*

**Proof.**

- ZF + Axiom of Choice implies Zorn's Lemma.

  We follow (HALMOS: NAIVE) and divide the proof into several steps.

  1. First we identify $(X, \leqslant)$ with some subset $\mathfrak{X}$ of $(\mathcal{P}(X), \subseteq)$. Indeed, we can identify
  $$x \in X \text{ with } s(x) := \{y \in X | \, y \leqslant x\}. \tag{70}$$
  Now the goal is to find a maximal set in $\mathcal{S}$. From now on we identify $(X, \leqslant)$ with $(\mathfrak{X}, \subseteq)$.

  2. Now let $f$ be a choice function on $X$. For any $A \subset \mathfrak{X}$, let $\hat{A} := \{x \in X | \, A \cup \{x\} \subseteq X\}$. Intuitively, $\hat{A}$ are all those elements in $X$ that can be appended to $A$ to form a longer chain. Now define
  $$g(A) := A \cup f\{\hat{A} - A\} \tag{71}$$
  and our goal is to find $A$ such that $g(A) = A$, which means $\hat{A} - A = \varnothing$ or $A$ is maximal.

  3. We say a subcollection $\mathfrak{T}$ of $\mathfrak{X}$ is a "tower" if and only if

     i. $\varnothing \in \mathfrak{J}$;

     ii. If $A \in \mathfrak{J}$ then $g(A) \in \mathfrak{J}$;

     iii. if $\mathfrak{C}$ is a chain in $\mathfrak{J}$, then $\cup_{A \in \mathfrak{C}} A \in \mathfrak{J}$.

  It is clear that $\mathfrak{X}$ is a tower. Now let $\mathfrak{J}_0$ be the intersection of all towers (and is therefore the smallest tower). We will prove that $\mathfrak{J}_0$ is a chain.

4. Now say $C \in \mathfrak{J}_0$ is comparable if and only if for every $A \in \mathfrak{J}_0$ different from $C$, either $A \subset C$ or $C \subset A$. It is clear that $\varnothing$ is comparable.

   Let $C$ be any fixed comparable set. Let $A \in \mathfrak{J}_0$ be a proper subset of $C$, then clearly $g(A)$ is also a proper subset of $C$.

5. Now consider the collection $\mathcal{U}$ of all the sets $A \in \mathfrak{J}_0$ with either $A \subseteq C$ or $g(C) \subseteq A$. We can prove that $\mathcal{U} = \mathfrak{J}_0$ through proving $\mathcal{U}$ is a tower. This implies, if $C$ is comparable, then so is $g(C)$. This implies that all the comparable sets in $\mathfrak{J}_0$ form a tower and consequently all the sets in $\mathfrak{J}_0$ are comparable. Therefore $\mathfrak{J}_0$ is a chain.

6. Finally, by iii in the definition of tower we have

$$A := \cup_{B \in \mathfrak{J}_0} B \in \mathfrak{J}_0 \tag{72}$$

   Since $g(A) \in \mathfrak{J}_0$, $g(A) \subseteq A$ and the proof ends.

- ZF + Zorn's Lemma implies Axiom of Choice.

  We prove this through considering all functions such that $\operatorname{dom} f \subseteq \mathcal{P}(X)$, $\operatorname{ran} f \subseteq X$ and $f(A) \in A$ for all $A \in \operatorname{dom} f$. We order these functions by extension and apply Zorn's Lemma. □

**Exercise 39.** Spot the mistake in the following proof of Zorn's Lemma without invoking Axiom of Choice:

Take any $x_0 \in X$. If $x_0$ is maximal we are done. Otherwise we can find $x_1 > x_0$ and $\{x_0, x_1\}$ is a chain. Now consider the collection $\mathcal{C} := \{Y \subseteq X \text{ is a chain}, x_0 \in Y\}$. Now let

$$Z := \cup_{Y \in \mathcal{C}} Y. \tag{73}$$

By assumption there is $a \in X$ which is an upper bound of $Z$. It is clear that $a$ must be maximal.

**Problem 2.** (HALMOS: NAIVE) Prove that each of the following is equivalent to Zorn's Lemma.

a) Every partially ordered set has a maximal chain;

b) Every chain in a partially ordered set is included in some maximal chain;

c) Every partially ordered set in which each chin has a least upper bound has a maximal element.

**Remark 59.** Recall that "maximal" (cannot get any bigger) is different from "greatest" (is the biggest).

### 4.3.3. Well-ordering

THEOREM 60. (WELL-ORDERING THEOREM) *Every set can be well-ordered.*

**Proof.** For the set $X$, we consider the collection of all of its subsets that could be well-ordered:

$$W := \{(A, \leqslant_A) | \, A \subseteq X, (A, \leqslant_A)] \text{ is well ordered}\}. \tag{74}$$

We order $W$ by "continuation":

A well ordered set $A$ is a continuation of a well-ordered set $B$ if the following are satisfied:

i. $B \subseteq A$;

ii. $B$ is a initial segment of $A$;

iii. The ordering on $B$ is the same as the ordering on $A$.

Then clearly every chain has a maximal element. Now apply Zorn's lemma, there is $Y \in W$ that is maximal. We see that it must hold that $Y = X$. □

**Exercise 40.** Fill in the details of the above proof.

**Exercise 41.** Prove that well-ordering theorem implies Axiom of Choice, and conclude that Axiom of Choice, Well-ordering theorem, and Zorn's Lemma are all equivalent.

## 4.4. Back to cardinal numbers

Our informal theory of cardinality leaves some important questions open. In particular, given any two sets $A, B$, are their cardinalities always comparable?

DEFINITION 61. (CARDINAL NUMBERS) *A cardinal number is an ordinal number $\lambda$, such that for all $\alpha < \lambda$, $\alpha$ has strictly smaller cardinality than $\lambda$.*

**Remark 62.** In other words, a cardinal number is an ordinal number that is strictly bigger than all its predecessors.

**Example 63.** Define

$$\omega_1 := \{\alpha \in W \mid \alpha \text{ is countable}\}. \tag{75}$$

Then $\omega_1$ is the smallest (with respect to cardinality) uncountable ordinal number. When seen as a cardinal number, we denote it by $\aleph_1$.

**Proof.** First we show $\omega_1$ is uncountable. Assume otherwise, then $\omega_1 = \alpha \in \omega_1$ which means $\omega_1$ is its own segment. Contradiction.

Now assume there is $\beta < \omega_1$ that is uncountable. By definition of ordinal numbers we have $\beta \in \omega_1$. Contradiction. □

**Exercise 42.** Prove that $\omega = \{\alpha \in W \mid \alpha \text{ is finite}\}$.

Now we can go on to denote the smallest (w.r.t. cardinality) ordinal number bigger (w.r.t. cardinality) than $\omega_1$ and $\omega_2$ by $\omega_3$ and define the third infinite cardinal number

$$\aleph_2 := \omega_2. \tag{76}$$

and then

$$\aleph_3, \omega_3,\ \aleph_4, \omega_4,\ \ldots, \aleph_\omega, \omega_\omega,\ \aleph_{\omega+1}, \omega_{\omega+1}, \ldots \tag{77}$$

THEOREM 64. *Assuming Axiom of Choice. Then any set $A$ corresponds to a cardinal number $\aleph_\alpha$ where $\alpha \in W$ is an ordinal number.*

**Proof.** Assuming AoC, then every set can be well-ordered and corresponds to an ordinal number. □

THEOREM 65. *Let $\alpha < \beta$ be ordinal numbers. Then*

$$\max(\aleph_\alpha, \aleph_\beta) = \aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \aleph_\beta. \tag{78}$$

**Exercise 43.** Prove that for any ordinal $\alpha$,

$$\aleph_\alpha + \aleph_\alpha = \aleph_\alpha; \qquad \aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha \tag{79}$$

and then use these to prove the above theorem.

# 5. More Exercises and Problems

## 5.1. Basic exercises

### 5.1.1. Ordering

**Exercise 44.** Consider the set $\mathbb{R}^2$. We define its ordering to be

$$(x_1, x_2) < (y_1, y_2) \Longleftrightarrow [(x_1 < x_2) \text{ or } (x_1 = x_2, y_1 < y_2)]. \tag{80}$$

A subset $E \subseteq \mathbb{R}^2$ is said to be "dense" with respect to this order if and only if for every pair $(x_1, x_2) < (y_1, y_2)$, there is $(z_1, z_2) \in E$ such that

$$(x_1, x_2) < (z_1, z_2) < (y_1, y_2). \tag{81}$$

Prove

a) $\mathbb{Q}^2$ is not dense for this ordering;

b) There is no countable dense subset for this ordering.

(Hint:[21] )

**Exercise 45.** Let an order on $\mathbb{N} \times \mathbb{N}$ be defined as

$$(a, b) \leqslant (x, y) \Longleftrightarrow (2\,a + 1)\,2^y \leqslant (2\,x + 1)\,2^b. \tag{82}$$

Prove that $\mathbb{N} \times \mathbb{N}$ with this order is not well-ordered. (Hint:[22] )

**Exercise 46.** Let $X$ be the set of all infinite sequeces $\{x_n\}_{n=1}^{\infty}$. Define

$$\{x_n\} < \{y_n\} \tag{83}$$

as "either $x_0 < y_0$; or $x_0 = y_0$ but $x_1 < y_1$; or $x_0 = y_0, x_1 = y_1$ but $x_2 < y_2$...". Is the set partially ordered? totally ordered? well-ordered? (Hint:[23] )

**Exercise 47.** Let $X$ be the set of all functions $f\colon \mathbb{R} \mapsto \mathbb{R}$.

a) Let $f \leqslant g$ be defined as $\forall x \in \mathbb{R}$, $f(x) \leqslant g(x)$;

b) Let $f \leqslant g$ be defined as $f = g$ or $\lim_{x \to \infty} (f(x)/g(x)) = 0$.

(Hint:[24] )

**Exercise 48.** Recall that a relation is defined through a set in $X \times Y$. Try to imagine what the set looks like when the relation is a "partial order", "total order", or "well order".

### 5.1.2. Similarity

**Exercise 49.** Are $[0, 1]$ and $[0, 100]$, both equaipped with the natural order, of the same order-type? Justify. (Hint:[25] )

**Exercise 50.** Are $[0, 1]$ and $[0, 100)$, both equipped with the natural order, of the same order-type? Justify. (Hint:[26] )

**Exercise 51.** Find two totally ordered sets $A, B$. Such that $A, B$ are not similar, but each is similar to a subset of the other. (Hint:[27] )

**Exercise 52.** Find a totally ordered set $A$ and a similarity mapping $f\colon A \mapsto A$ such that $\forall x \in A$, $f(x) \neq x$. (Hint:[28] )

21. $(1, \sqrt{2}) < (2, \sqrt{2})$; Any $y_0 \in \mathbb{R}$, any dense set must have at least one element of the form $(x, y_0)$.
22. The ordering is the same as $\frac{2\,a+1}{2^b} \leqslant \frac{2\,x+1}{2^y}$ where the $\leqslant$ is the usual "less than or equal".
23. Totally orderred. Now well-ordered since $\mathbb{R}$ with usual ordering is not well-ordered.
24. partial order; partial order.
25. Yes. $f(x) = 100\,x$.
26. No. Consider $f(1)$.
27. $(0, 1)$ and $[0, 1]$.
28. $\mathbb{Z} \mapsto \mathbb{Z}$, $+1$.

**Exercise 53.** Prove that, a totally ordered set is well-ordered if and only if there is no infinite descending sequence. (Hint:[29] )

**Exercise 54.** Let $A, B$ be disjoint totally ordered sets. Prove

a) If $A, B$ are similar then $\langle A \cup B \rangle$ and $\langle B \cup A \rangle$ are similar;

b) $\langle A \cup B \rangle$ and $\langle B \cup A \rangle$ being similar does not imply $A, B$ are similar. (Hint:[30] )

### 5.1.3. Ordinal numbers

**Exercise 55.** Re-order $\mathbb{N}$ to obtain $\omega^3$. (Hint:[31] )

**Exercise 56.** Prove the following.

a) $\omega + 1 \neq \omega + 2 \neq \omega + 3$; (Hint:[32] )

b) $1 + \omega = 2 + \omega = 3 + \omega$;

c) $\omega + 5 + \omega = \omega \cdot 2$; (Hint:[33] )

d) $3 + \omega + \omega = \omega \cdot 2$.

**Exercise 57.** Find a set $E$ of real numbers with natural ordering such that $\mathrm{Ord}(E) = \omega^3 := \omega \cdot \omega \cdot \omega$. (Hint:[34] )

## 5.2. More exercises

**Exercise 58.** Let $X$ be well-ordered. Prove that it has least upper bound property, that is if $A \subseteq X$ has an upper bound, that is there is $b \in X$ such that $a \leqslant b$ for all $a \in A$ then there is a least upper bound, that is an upper bound $a_0$ such that any other upper bound $a > a_0$. Does the converse hold? (Hint:[35] )

**Exercise 59.** Re-order $[0, 1]$ to have the same order-type as $[0, 100)$ equipped with the natural order. (Hint:[36] )

**Exercise 60.** Let $A$ be a countable totally ordered set. Then there is a subset $B \subseteq \mathbb{Q}$ such that $A$ and $B$ (with the natural order of $\mathbb{Q}$) are similar. (Hint:[37] )

**Exercise 61.** Let $A$ be totally ordered. If every $B \subseteq A$ has a first element, and a last element, then $A$ is finite. (Hint:[38] )

**Exercise 62.** Prove that

$$\sum_{n < \omega} n = \omega. \tag{84}$$

**Exercise 63.** Try to define an appropriate ordering on the set $A \times B$ so that

$$\mathrm{Ord}(A \times B) = \mathrm{Ord}(A) \cdot \mathrm{Ord}(B). \tag{85}$$

**Exercise 64.** Prove that

$$(\alpha > 0, \beta > \gamma) \Longrightarrow \alpha \cdot \beta > \alpha \cdot \gamma \tag{86}$$

then conclude

$$(\alpha > 0, \alpha \cdot \beta = \alpha \cdot \gamma) \Longrightarrow \beta = \gamma. \tag{87}$$

29. $a_1 > a_2 > a_3 > \cdots$ would be a non-empty subset with no least element.

30. Try $B = \langle A \cup A \rangle$.

31. $\mathbb{N}^3 \mapsto \mathbb{N}$: $(1,1,1), (1,1,2), (1,2,1), (2,1,1), \ldots$.

32. Assume otherwise. Then there is a similarity mapping between $1, 2, \ldots, \omega$ and $1, 2, \ldots, \omega, \omega + 1$. Clearly $\omega$ has to be mapped to $\omega + 1$. Consider which number could be mapped to $\omega$.

33. $1, 2, \ldots., \omega, \omega + 1, \ldots, \omega + 4, a_1, a_2, \ldots.$ Absorb the "5" into the 2nd $\omega$.

34. Note that $0, 1 - 1/2, 1 - 1/3, \ldots; 1, 2 - 1/2, 2 - 1/3, \ldots; 2, 3 - 1/2, 3 - 1/3, \ldots$ gives $\omega^2$.

35. $\mathbb{R}$ has least upper bound property.

36. Suffices to re-order to $[0, 1)$. We write $[0, 1] = \{0\} \cup \left[\cup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right]\right]$. We map 0 to $0 \in [0, 1)$. Then re-order each $\left(\frac{1}{n+1}, \frac{1}{n}\right]$ to $\left[\frac{1}{n+1}, \frac{1}{n}\right)$.

37. List $A = \{a_1, a_2, \ldots\}$. Map $a_1$ to any $r_1 \in \mathbb{Q}$. If $a_2 > (<) a_1$, map it to any $r_2 > (<) r_1$. Map $a_3$ to $r_3$ such that the order relation of $r_1, r_2, r_3$ is the same as that of $a_1, a_2, a_3$, and so on.

38. Let $a_1, b_1$ be the first and last elements of $A$. Let $a_2, b_2$ be the first and last elements of $A - \{a_1, b_1\}$. And so on.

**Exercise 65.** Prove that

a) $(\alpha > 0, \beta > 0) \Longrightarrow \alpha \cdot \beta \geqslant \alpha, \alpha \cdot \beta \geqslant \beta$;

b) $(\alpha > 0, \beta > 1) \Longrightarrow \alpha \cdot \beta > \alpha$.

## 5.3. Problems

**Problem 3.** We say a total ordering of a set $A$ is "dense" if for any $a \neq b$, there is $c$ in between. Prove that any densely ordered set without a first and last element is similar to $\mathbb{Q}$.[39]

**Problem 4.** Prove that, every denumerable totally ordered set is similar to a subset of $\mathbb{Q}$ with natural ordering.

**Problem 5.** Let $\alpha, \beta, \gamma$ be ordinal numbers. Prove

a) $\alpha + \beta = \alpha + \gamma \Longrightarrow \beta = \gamma$;

b) If $\alpha > \beta$ then there is a unique $\gamma$ such that $\alpha = \beta + \gamma$;

c) If $\alpha \geqslant \beta, \alpha_1 \geqslant \beta_1$, then $\alpha + \alpha_1 \geqslant \beta + \beta_1$. What about strict inequalities?

39. For proof see Hausdorff, *Set Theory*, Chapter 3, §11, Theorem IV.